Homotopy Theory of Differential Graded Modules and Adjoints of Restriction of Scalars

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Abstract

This study constructs two different model structures called projective and injective model on the category of differential graded modules over a differential graded ring and also provides an explicit description of fibrant and cofibrant objects for these models. The constructions are based on the concept and properties of semi-projective and semi-injective modules and other kinds of projectivity and injectivity in the category of differential graded modules.

Also an analysis of behavior of functors; restriction, extension and co-extension of scalars is given. Furthermore, some conditions under which an adjunction becomes a Quillen pair and a Quillen pair becomes a Quillen equivalence are described. Additionally, a relationship between restriction and co-extension for compact Lie groups is discovered.

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Contents

Introduction

The ideas of topological homotopy theory occur in different parts of mathematics such as chain complexes, simplicial sets and groupoids. An abstract approach toward studying geometric objects and concepts such as cylinders, path space structures and deformation enables us to develop and unify more examples as well as discovering logical interdependence of these ideas.

The abstract approach to find a common structure in different settings can be considered as abstract homotopy theory. There were various attempts, to find a axiomatic structure, among which Quillen model structure is the most widespread one and has often been considered as the basic abstract homotopy theory. To study more about abstract homotopy theory [7] can be considered as a start point but [31] provides a useful note.

By definition a model category is a category with three classes of morphisms called weak equivalence, fibration and cofibration satisfying some simple axioms which provide a machinery of homotopy theory. However, checking these axioms in different settings is more likely to be not an easy task.

For the category of non-negative chain complexes over a ring K, $Ch(K)_{\geq 0}$ a model structure has been given in [9]. In addition, Hovey in [19] developed two model structures on the category of chain complexes over a ring K and showed that there are two model structures on $Ch(K)$ known as the projective and the injective model. The number of results from [19] are summarized in the following table.

Model	Weak equivalences	Fibration	Cofibration	Cofibrantly Generated
Projective	quasi-isomorphism	surjections	sub-class of injections	yes
Injective	quasi-isomorphism	sub-class of surjections	injections	yes

In fact for the projective model, a map is a cofibration if and only if it is a dimensionwise split injection with cofibrant cokernel and for the injective model, a map is a fibration if and only if it is a dimensionwise split surjection with fibrant kernel.

Now a question may rise about how a similar table could be filled for $\mathcal{DGM}(R)$ when R is a differential graded algebra and $\mathcal{DGM}(R)$ denotes the category of differential graded $R-$modules. Schwede and Shipley in [34] stated the following theorem which may be employed to fill some parts of the table for $\mathcal{DGM}(R)$

Theorem. *[34, 4.1] Let $\mathcal{C}$ be a cofibrantly generated, monoidal model category. Assume further that every object in $\mathcal{C}$ is small related to the whole category and that $\mathcal{C}$ satisfies the monoid axiom.*

1. *Let R be a monoid in $\mathcal{C}$. Then the category of left R–modules is a cofibrantly generated model category.*
2. *Let R be a commutative monoid in $\mathcal{C}$. Then the category of left R–modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*

Looking at the DGA R as a monoid in the category of chain complexes over the ring of degree zero elements of R, denoting by $Ch(R_0)$, and having the previous theorem in hand, the above table shows that $\mathcal{DGM}(R)$ is a model category and there is a model structure on it corresponding to the projective model structure on $Ch(R_0)$. However regarding the injective model on the category of chain complexes, there are some obstructions because it is not a monoidal model structure.

Moreover, one can employ the next theorem, in which $\mathcal{A}$ is a DG-category and $\mathcal{C}(\mathcal{A})$ is the category of chain complexes of $\mathcal{A}$, to define two model structures on

$\mathcal{DGM}(R)$.

Theorem. *[24, 3.2] The category $\mathcal{C}(\mathcal{A})$ admits two structures of Quillen model category whose weak equivalences are the quasi-isomorphisms:*

i. The projective structure, whose fibrations are the epimorphisms. For this structure, each object is fibrant and an object is cofibrant if and only if it is a cofibrant DG-module.

ii. The injective structure, whose cofibrations are the monomorphisms. For this structure, each object is cofibrant and an object is fibrant if and only if it is a fibrant DG-module.

Note that a direct proof for the previous theorem is not provided in [24] rather it refers to [19, 2.3] as a source of techniques whereas they may not be working in the case of differential graded modules over a DGA and some necessary and not obvious modifications may be inevitable. However a proof for a similar statement to part (i) is provided in Theorem [5, 3.3]. Nonetheless, [24, 3.2] and [34, 4.1] do not present the fibrant and cofibrant objects explicitly.

This study aims to define two model structures on $\mathcal{DGM}(R)$ and provide an explicit expression for fibrations and cofibration as well as fibrant and cofibrant objects in the case of the projective and the injective structures. Moreover, analyzing the relation between two model categories is another scope of this study.

Outline of the Thesis The structure of my thesis is as follows.

Chapter 1 is an intensive review of the theory of Quillen model categories and contains all relevant material especially cofibrantly generated model categories and the small object argument.

A lucid exposition of the category of differential modules over a DGA is provided in chapter 2. The first four sections mostly prepare essential notations and definitions as well as some well known results. In the rest of this chapter, some important objects such as semi-free, semi-projective and semi-injective modules, playing a vital role in the rest of the thesis, are introduced and the properties of these objects are explained.

It is worth mentioning that most material for the last five sections of chapter 2 is taken from [4] and slight modifications are done in the line of this research.

Most of the main results of this study are given in chapter 3. In fact, the following theorem is proved in this chapter. For the definition of semi-projective and semi-injective modules see 2.50 and 2.63.

Theorem. *The category $\mathcal{DGM}(R)$ admits two structures of Quillen model category whose weak equivalences are the quasi-isomorphisms:*

i. The cofibrantly generated projective structure, whose fibrations are the surjections. For this structure, every object is fibrant and an object is cofibrant if and only if it is a semi-projective DG-module . In addition, cofibrations are the injections with semi-projective cokernel.

ii. The injective structure, whose cofibrations are the injections. For this structure, every object is cofibrant and an object is fibrant if and only if it is a semi-injective DG-module. In addition, fibrations are the surjections with semi-injective kernel.

Moreover, in section 3.4 a criterion for determining semi-injective modules will be given. Section 3.6 deals with both constructive and non-constructive methods to find a semi-projective and semi-injective resolution for a DG-module. Furthermore, the last section of this chapter shows how to apply the cotorsion theory for defining the projective and the injective models.

Chapter 4 investigates the behavior of restriction, extension and co-extension of scalars functors between two model categories and finds the relation among these functors and two other functors which will be introduced at the beginning of this chapter. Additionally, some necessary and sufficient conditions to govern the behavior of these functors at the derived level are determined. In fact, these conditions explain when an adjunction is a Quillen pair and when a Quillen pair is a Quillen equivalences. Moreover, in section 4.4 we introduce a slightly modified definition of Gorenstein rings and provide varieties of examples and finally the last section of chapter 4 deals with the relationship between restriction and co-extension for a compact Lie groups.

1
Model Categories

The idea of inverting a certain class of morphisms or objects happens quite often in mathematics. For a given category and a class of morphisms one may like to consider this class of morphisms as isomorphisms. It is possible to build a new category with the objects remaining the same and formally invert the class of morphisms. This process is called localization. However, this kind of localization sometimes is not a locally small category [23, 7.1].
The model categories, first introduce by Quillen [32], are categories with necessary structures to build a locally small category, called homotopy category, such that the homotopy category is equivalent to its localization.
In this chapter, an overview of model categories will be given and most of the material can be found in [19], [9], [18] and [15].

1.1 Pre-requirements and General Definitions

Definition 1.1. Let $\mathcal{C}$ be a category and $Obj(\mathcal{C})$ and $Mor(\mathcal{C})$ denote the class of objects and morphisms in $\mathcal{C}$ respectively.

i. For $f, g \in Mor(\mathcal{C})$, f is retract of g if and only if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A' \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & B' \arrow[r] & B \end{tikzcd}$$

such that the composition of the horizontal maps are the identity on A and B.

ii. Let α, β be functors $Mor(\mathcal{C}) \longrightarrow Mor(\mathcal{C})$. Then the pair (α, β) is a functorial factorization, if $f = \beta(f) \circ \alpha(f)$ for all $f \in Mor(\mathcal{C})$.

iii. Suppose $f, g, i, p \in Mor(\mathcal{C})$, a lift for the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

is a morphism $h : B \longrightarrow C$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

commutes. i is said to have left lifting property (LLP) with respect to p and p is said to have right lifting property (RLP) with respect to i.

Definition 1.2. [18] A model category is a category $\mathcal{M}$ with three closed subclasses of morphisms that include identities: weak equivalences $(\xrightarrow{\simeq})$, fibrations $(\twoheadrightarrow)$, and cofibrations $(\hookrightarrow)$. These subclasses must also satisfy axioms **MC1-MC5**.

A trivial fibration (cofibration) is a morphism which is a fibration (cofibration) and a weak equivalence.

MC1 $\mathcal{M}$ is complete and cocomplete, i.e. limits and colimits exist in $\mathcal{M}$.

MC2 If $f, g \in Mor(\mathcal{M})$ such that $g \circ f \in Mor(\mathcal{M})$ and two of the three maps are weak equivalences, then so is the third. This is called two of three property.

MC3 If f is a retract of g and g is a weak equivalence, fibration or cofibration then so is f, respectively.

MC4 If $f, g, i, p \in Mor(\mathcal{M})$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

commutes, and i is a cofibration (trivial cofibration) and p is a trivial fibration (fibration), then there exists a lift $h : B \longrightarrow C$.

MC5 If $f \in Mor(\mathcal{M})$ then there are functorial factorizations (α, β) and (γ, δ) such that $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

Remark 1.3. A model category was originally called a closed model category to emphasize that it has enough structures to guarantee that any two classes of morphisms determine the third one. Note that some definitions require only the finite limits and colimits in **MC1** and the functoriality in **MC5** is not a mandatory condition.

Remark 1.4. **MC1** guarantees that a unique initial object $\emptyset$ and a unique terminal object $*$ exist in $\mathcal{M}$.

Definition 1.5. For $X \in Obj(\mathcal{M})$, if the unique map $\emptyset \longrightarrow X$ is a cofibration then X is a cofibrant object and if the unique map $X \longrightarrow *$ is a fibration then X is a fibrant object.

Proposition 1.6. *[19] Suppose $\mathcal{M}$ is a model category.*

i. The fibrations (trivial fibrations) in $\mathcal{M}$ are the maps which have the right lifting property with respect to trivial cofibrations (cofibrations).

ii. The cofibrations (trivial cofibrations) in $\mathcal{M}$ are the maps which have the left lifting property with respect to trivial fibrations (fibrations).

iii. The fibrations (trivial fibrations) are stable under pullback and the cofibrations (trivial cofibrations) are stable under pushout.

1.2 Homotopy Category

To construct the homotopy category of a model category, some new tools are needed. These tools are in fact the generalization of the classical definitions in the category of topological spaces and chain complexes.

Constructive method Let $\mathcal{M}$ be a model category.

Definition 1.7. [9] In the category $\mathcal{M}$

i. A path object of Y, is any object P_Y such that there is a commutative diagram

$$\begin{tikzcd} & P_Y \arrow[dr, "p"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\Delta"] & & Y \sqcap Y \end{tikzcd}$$

where Δ is the diagonal map and i is a weak equivalence. A path object P_Y is a *good* path object if p is a fibration and is a *very good* path object if p is a fibration and i is a cofibration.

ii. The maps $f, g : X \longrightarrow Y$ are right homotopic, $f \sim_r g$, if for some path object P_Y of Y, there exists a map $H : X \longrightarrow P_Y$ such that the diagram

$$\begin{tikzcd} & P_Y \arrow[d, "p"] \\ X \arrow[ur, dotted, "H"] \arrow[r, "f \sqcap g"'] & Y \sqcap Y \end{tikzcd}$$

commutes. The map H is called a right homotopy from f to g. Also, if P_Y is a (very) good path object then H is a (very) good right homotopy.

iii. A cylinder object of X is any object C_X such that there is a commutative diagram

$$\begin{tikzcd} & C_X \arrow[dr, "p"] & \\ X \sqcup X \arrow[ur, "i"] \arrow[rr, "\triangledown"'] & & X \end{tikzcd}$$

where $\triangledown$ is the folding map and p is a weak equivalence. A cylinder object C_X is a *good* cylinder object if i is a cofibration and is a *very good* cylinder object if i is a cofibration and p is a fibration.

iv. The maps $f, g : X \longrightarrow Y$ are left homotopic, $f \sim_l g$, if for some cylinder object C_X of X, there exists a map $H : C_X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "f \sqcup g"] \arrow[d, "i"'] & Y \\ C_X \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

commutes. The map H, is called a left homotopy from f to g. Also, if C_X is a (very) good cylinder object then H is a (very) good left homotopy.

v. The maps $f, g : X \longrightarrow Y$ are homotopic $f \sim g$, if they are both left and right homotopic.

vi. The map $f : X \longrightarrow Y$ is a homotopy equivalence if there exists a map $g : Y \longrightarrow X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

Lemma 1.8. *[9] Suppose $\mathcal{M}(X, Y)$ is the set of morphisms from X to Y in $\mathcal{M}$ and $f, g \in \mathcal{M}$.*

i. The relation $\sim_r$ is an equivalence relation on $\mathcal{M}(X, Y)$ if Y is a fibrant object.

ii. The relation $\sim_l$ is an equivalence relation on $\mathcal{M}(X, Y)$ if X is a cofibrant object.

iii. If Y is a fibrant object and $f \sim_r g$, then $f \sim_l g$.

iv. If X is a cofibrant object and $f \sim_l g$, then $f \sim_r g$.

Theorem 1.9. *Given a map $f : X \longrightarrow Y$ in $\mathcal{M}$ such that X and Y are fibrant-cofibrant objects then, f is a weak equivalence if and only if it is a homotopy equivalence.*

Definition 1.10. For every object X in $\mathcal{M}$ a fibrant replacement of X is an object RX in the commutative diagram below which always exists due to **MC5**.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\simeq"] & & * \\ & RX \arrow[ur, two heads] & \end{tikzcd}$$

Similarly, a cofibrant replacement of X is an object QX in the diagram

$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[dr, hook] & & X \\ & QX \arrow[ur, two heads, "\simeq"] & \end{tikzcd}$$

Lemma 1.11. *For every map $f : X \longrightarrow Y$ there exists $f^* : QRX \longrightarrow QRY$ such that f is a weak equivalence if and only if f^* is a weak equivalence. The map f^* is unique up to homotopy.*

Let $\mathcal{M}_{cf}$ be the category of all objects from $\mathcal{M}$ which are both fibrant and cofibrant and with the morphisms same as $\mathcal{M}$. In addition, suppose $\mathcal{M}/\sim$ denotes the category with the same objects as $\mathcal{M}_{cf}$ but let the morphisms be the quotient of the morphisms in $\mathcal{M}$ by homotopy.

Theorem 1.12. *[9] The fibrant-cofibrant replacement map* $QR : \mathcal{M} \longrightarrow \mathcal{M}_{cf}/\sim$ *defined by* $X \longmapsto QRX$ *for every* $X \in \mathcal{M}$ *and* $f \longmapsto [f^*]$ *for every* $f \in \mathcal{M}(X,Y)$ *is a functor.*

Definition 1.13. The homotopy category of $\mathcal{M}$ is the category $Ho(\mathcal{M})$ where

$$Obj(Ho(\mathcal{M})) = Obj(\mathcal{M})$$

and

$$Ho(\mathcal{M})(X,Y) = \mathcal{M}(QRX, QRX)/\sim$$

Theorem 1.14. *Let* $H_{\mathcal{M}} : \mathcal{M} \longrightarrow Ho(\mathcal{M})$ *be defined by* $X \longmapsto X$ *for all* $X \in \mathcal{M}$ *and* $f \longmapsto [QR(f)\,]$ *for all* $f \in Mor(\mathcal{M})$*. Then* $H_{\mathcal{M}}$ *is a functor. Furthermore,* $H_{\mathcal{M}}(f)$ *is an isomorphism if and only if* f *is a weak equivalence.*

Non-constructive method The previous statements in this section give us a constructive approach towards defining a homotopy category. Now, by defining a localization of a category, a non-constructive approach will be introduced and we will see that the homotopy category constructed from a model category is isomorphic to the localization of the model category with respect to the class of weak equivalences.

Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a class of morphisms of $\mathcal{C}$.

Definition 1.15. [23, 7.1.1] A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is the data of a large category $\mathcal{W}^{-1}\mathcal{C}$ and a functor $F : \mathcal{C} \longrightarrow \mathcal{W}^{-1}\mathcal{C}$ satisfying

i. $F(w)$ is an isomorphism for all $w \in \mathcal{W}$

ii. For any large category $\mathcal{D}$ and any functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ such that $G(w)$ is an isomorphism for all $w \in \mathcal{W}$, there exists a functor $U : \mathcal{W}^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{D} \\ \mathcal{W}^{-1}\mathcal{C} \arrow[ur, "U"'] & \end{tikzcd}$$

commutes

iii. If U_1, U_2 are two objects of $\mathcal{D}^{\mathcal{W}^{-1}\mathcal{C}}$ then the natural map

$$\mathcal{D}^{\mathcal{W}^{-1}\mathcal{C}}(U_1, U_2) \longrightarrow \mathcal{D}^{\mathcal{C}}(U_1 \circ F, U_2 \circ F)$$

is bijective.

Theorem 1.16. *[9] The functor $H_{\mathcal{M}}$ is a localization of $\mathcal{M}$ with respect to the class of weak equivalences.*

Hence, by universal property of localization, $Ho(\mathcal{M}) \cong \mathcal{W}^{-1}\mathcal{M}$ where $\mathcal{W}$ is the class of weak equivalences.

1.3 Derived Functors and Quillen Functors

For a model category $\mathcal{M}$, define $Ho(\mathcal{M})$ to be its homotopy category and

$$H_{\mathcal{M}} : \mathcal{M} \longrightarrow Ho(\mathcal{M})$$

to be the homotopy functor and $1_{H_{\mathcal{M}}}$ to be the identity natural transformation on $H_{\mathcal{M}}$.

Definition 1.17. Let $F : \mathcal{M} \longrightarrow \mathcal{D}$ be a functor from a model category to a category. Then a left derived functor of F is a pair of a functor and a natural transformation (LF, l) such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "H_{\mathcal{M}}"'] & \arrow[d, phantom, "\Uparrow l"] & \mathcal{D} \\ & Ho(\mathcal{M}) \arrow[ur, "LF"'] & \end{tikzcd}$$

commutes and if (G, l') is any other such pair, there exists a natural transformation $t : G \longrightarrow LF$ such that $l \circ (t \circ 1_{H_{\mathcal{M}}}) = l'$.

Similarly, a right derived functor of F is a pair (RF, r) such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "H_{\mathcal{M}}"'] & \arrow[d, phantom, "\Downarrow r"] & \mathcal{D} \\ & Ho(\mathcal{M}) \arrow[ur, "RF"'] & \end{tikzcd}$$

commutes and if (G, r') is any other such pair, there exists a natural transformation $t : RF \longrightarrow G$ such that $(t \circ 1_{H_{\mathcal{M}}}) \circ r = r'$.

Remark 1.18. As a result of the universal property, a left or right derived functor is unique up to a unique isomorphism. Hence we refer just to the left or right derived functor. Note that [18, 8.4.1] describes the left derived functor as the closest functor to F on the left and right derived functor as the closest functor to F on the right.

Definition 1.19. Let $F : \mathcal{M} \longrightarrow \mathcal{M}'$ be a functor between two model categories. Then the total left derived functor $(\mathbf{L}F, \mathbf{l})$ is the left derived functor of composition $H_{\mathcal{M}'} \circ F : \mathcal{M} \longrightarrow Ho(\mathcal{M}')$. Similarly, the total right derived functor $(\mathbf{R}F, \mathbf{r})$ is the right derived functor of $H_{\mathcal{M}'} \circ F$.

One can think of the total derived functor as a good extension of a functor between model categories to their homotopy category. However, this good extension sometimes doesn't exist. The next theorem gives some sufficient conditions under which the total derived functor exists.

Theorem 1.20. *[18] Let $F : \mathcal{M} \longrightarrow \mathcal{D}$ be a functor from a model category to a category.*

i. If F sends trivial cofibrations between cofibrant objects to isomorphisms in $\mathcal{D}$, then the left derived functor of F, (LF, l) exists and for a cofibrant object X, the natural map l_X is an isomorphism.

ii. If F sends trivial fibrations between fibrant objects to isomorphisms in $\mathcal{D}$, then the right derived functor of F, (RF, r) exists and for a fibrant object Y, the natural map r_Y is an isomorphism.

Theorem 1.21. *[9] Let $\mathcal{M}$ and $\mathcal{M}'$ be model categories and*

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}'$$

be an adjoint pair where F is a left adjoint to G. If F preserves cofibrations and G preserves fibrations, then

$$Ho(\mathcal{M}) \underset{\boldsymbol{R}G}{\overset{\boldsymbol{L}F}{\rightleftarrows}} Ho(\mathcal{M}')$$

are adjoint. In addition, if for every cofibrant object $X \in \mathcal{M}$ and every fibrant object $Y \in \mathcal{M}'$, $F(X) \longrightarrow Y$ is a weak equivalence if and only if its adjoint morphism $X \longrightarrow G(Y)$ is a weak equivalence, then $\boldsymbol{L}F$ and $\boldsymbol{R}G$ are equivalences of categories.

Theorems 1.20 and 1.21 describe the behavior of an adjunction between two model categories and show that the structure of a model category will be preserved under some conditions which lead us to the following definition.

Definition 1.22. Let $\mathcal{M}$ and $\mathcal{M}'$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}'$$

be an adjoint pair where F is a left adjoint to G and ϕ is the natural isomorphism of adjunction.

i. F is a left Quillen functor if it preserves cofibrations and trivial cofibrations.

ii. G is a right Quillen functor if it preserves fibrations and trivial fibrations.

iii. (F, G, ϕ) is a Quillen adjunction if F is a left Quillen functor. The pair (F, G) is called Quillen pair.

Remark 1.23. Ken Brown's lemma [19, 1.1.12] implies that every left Quillen functor preserves weak equivalences between cofibrant objects and similarly, every right Quillen functor preserves weak equivalences between fibrant objects. Moreover by Lemma [19, 1.3.4], (F, G, ϕ) is a Quillen adjunction if and only if G is a right Quillen functor.

Theorem 1.24. *[18] Let $\mathcal{M}$ and $\mathcal{M}'$ be model categories and*

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}'$$

be a Quillen pair. If X is a cofibrant object of $\mathcal{M}$ and Y is a fibrant object of $\mathcal{M}'$, then the isomorphism

$$\mathcal{M}'(F(X), Y) \cong \mathcal{M}(X, G(Y))$$

induces an isomorphism

$$\mathcal{M}'(F(X), Y)/\sim \cong \mathcal{M}(X, G(Y))/\sim .$$

Now, we give the definition of functors which look like isomorphisms between two model categories.

Definition 1.25. A Quillen adjunction $(F, G, \phi) : \mathcal{M} \longrightarrow \mathcal{M}'$ is called a Quillen equivalence if and only if for all cofibrant objects $X \in \mathcal{M}$ and fibrant objects $Y \in \mathcal{M}'$, a map $f : F(X) \longrightarrow Y$ is a weak equivalence in $\mathcal{M}'$ if and only if its adjoint morphism $\phi(f) : X \longrightarrow G(Y)$ is weak equivalence in $\mathcal{M}$.

Combining Theorem 1.21 and Definition 1.25 results in the following proposition.

Proposition 1.26. *A Quillen adjunction $(F, G, \phi) : \mathcal{M} \longrightarrow \mathcal{M}'$ is a Quillen equivalence if and only if*

$$Ho(\mathcal{M}) \underset{\boldsymbol{R}G}{\overset{\boldsymbol{L}F}{\rightleftarrows}} Ho(\mathcal{M}')$$

is an adjoint equivalence of categories.

The next Lemma is very useful to check if a given Quillen adjunction is a Quillen equivalence. Recall that a functor *reflects* a property of morphisms if, given a morphism f, if $F(f)$ has the property so does f.

Lemma 1.27. *[19] Suppose $(F, G, \phi) : \mathcal{M} \longrightarrow \mathcal{M}'$ is a Quillen adjunction. The following are equivalent:*

i. *(F, G, ϕ) is a Quillen equivalence.*

ii. *F reflects weak equivalence between cofibrant objects and, for every fibrant object Y, the map $FQG(Y) \longrightarrow Y$ is a weak equivalence.*

iii. *G reflects weak equivalence between fibrant objects and, for every cofibrant object X, the map $X \longrightarrow GRF(X)$ is a weak equivalence.*

1.4 Cofibrantly Generated Model Categories

Most often it is a quite difficult task to show that a category admits a model category structure. In this section, the number of topics, helping us to minimize things to check, are introduced based on the material of [19, 2.1].

Definition 1.28. Let $\mathcal{C}$ be a category with all small colimits and I be a collection of morphisms in $\mathcal{C}$. In addition, suppose κ is a cardinal and λ is an ordinal.

i. The ordinal λ is $\kappa-$filtered if it is a limit ordinal and if $A \subseteq \lambda$ and $|A| \leq \kappa$ then $Sup A < \lambda$

ii. A $\lambda-$sequence in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \longrightarrow \mathcal{C}$ commonly written as

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_\beta \longrightarrow \ldots$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\varinjlim_{\beta<\gamma} X_\beta \longrightarrow X_\gamma$$

is an isomorphism. We refer to the map $X_0 \longrightarrow \varinjlim_{\beta<\lambda} X_\beta$ as the composition of the $\lambda-$sequence. If every map $X_\beta \longrightarrow X_{\beta+1}$ for $\beta + 1 < \lambda$ is in I the composition $X_0 \longrightarrow \varinjlim_{\beta<\lambda} X_\beta$ is called a transfinite composition of maps of I.

iii. An object P in $\mathcal{C}$ is $\kappa-$small relative to I if, for all $\kappa-$filtered ordinals λ and all $\lambda-$sequences of maps of I, the canonical map

$$\varinjlim_{\beta<\lambda} \mathcal{C}(P, X_\beta) \longrightarrow \mathcal{C}(P, \varinjlim_{\beta<\lambda} X_\beta)$$

is an isomorphism. Moreover, P is small (finite) relative to I if it is $\kappa-$small relative to I for some (finite) κ. We say P is (finite) small if it is (finite) small relative to $\mathcal{C}$ itself.

Example 1.29. Suppose R is a ring then the finitely presented $R-$modules are finite.

Definition 1.30. Let I be a class of maps in a category $\mathcal{C}$

i. A map is $I-$injective if it has the right lifting property with respect to every map in I. The class of $I-$injective maps is denoted by I*–inj*.

ii. A map is $I-$projective if it has the left lifting property with respect to every map in I. The class of $I-$projective maps is denoted by I*–proj*.

iii. A map is an $I-$fibration if it has the right lifting property with respect to every map in I–$proj$. The class of $I-$fibrations is the class of $(I$–$proj)$–inj and is denoted by I–fib.

iv. A map is an $I-$cofibration if it has the left lifting property with respect to every map in I–inj. The class of $I-$cofibrations is the class of $(I$–$inj)$–$proj$ and is denoted by I–cof.

If $\mathcal{M}$ is a model category and, I is the class of cofibrations, then I–inj is the class of trivial fibrations and, I–$cof = I$. Dually, if I is the class of fibrations, then I–$proj$ is the class of trivial cofibrations and, I–$fib = I$.

Note that in any category, $I \subseteq I$–cof and $I \subseteq I$–fib. Also, $(I$–$cof)$–$inj = I$–inj and $(I$–$fib)$–$proj = I$–$proj$. Furthermore, if $I \subseteq J$ then I–$inj \supseteq J$–inj and I–$proj \supseteq J$–$proj$. Hence, I–$cof \subseteq J$–cof and I–$fib \subseteq J$–fib.

Definition 1.31. Let I be a *set* of maps in a category $\mathcal{C}$ containing all small colimits. A relative I-cell complex is a transfinite composition of pushouts of elements of I. The collection of I-cell complexes is denoted by I–$cell$.

Lemma 1.32. *Let I be a* set *of maps in a category $\mathcal{C}$ containing all small colimits, then*

i. I–$cell \subseteq I$–cof.

ii. I–$cell$ is closed under transfinite compositions.

iii. Any pushout of coproducts of maps of I is in I–$cell$

Theorem 1.33 (Small object argument)**.** *Suppose I is a set of maps in a category $\mathcal{C}$, containing all small colimits, such that the domains of the maps of I are small relative to I–$cell$. Then there is a functorial factorization (α, β) on $\mathcal{C}$ such that for all morphisms f in $\mathcal{C}$, the map $\alpha(f)$ is in I–$cell$ and the map $\beta(f)$ is in I–inj.*

The small object argument gives us a strong tool to construct model categories. Now we define a cofibrantly generated model category and show how to construct cofibrantly generated model categories.

Definition 1.34. Suppose $\mathcal{M}$ is a model category. We say that $\mathcal{M}$ is cofibrantly generated if there are sets I and J of maps such that,

i. The domain of the maps of I and J are small relative to $I{-}cell$ and $J{-}cell$ respectively.

ii. The class of fibrations is J–inj and the class of trivial fibrations is I–inj.

Proposition 1.35. *Suppose $\mathcal{M}$ is a cofibrantly generated model category, with generating cofibration I and generating trivial cofibrations J. Then*

i. The cofibrations form the class I–cof.

ii. Every cofibration is a retract of a member of I–$cell$.

iii. The domains of I are small relative to cofibrations.

iv. The trivial cofibrations form the class J–cof.

v. Every trivial cofibration is a retract of a member of J–$cell$.

vi. The domains of J are small relative to the trivial cofibrations.

The next theorem plays an important role in defining the projective model on the category of differential graded modules. In fact, it provides an alternative definition for a cofibrantly generated model category.

Theorem 1.36. *Suppose $\mathcal{C}$ is a category with all small limits and colimits. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there exists a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if following conditions are satisfied.*

i. The subcategory $\mathcal{W}$ has the two of three property and is closed under retract.

ii. The domains of I are small relative to I–$cell$.

iii. The domains of J are small relative to J–$cell$.

iv. J–$cell \subseteq \mathcal{W} \cap I$–$cof$.

v. I–$inj \subseteq \mathcal{W} \cap J$–$inj$.

vi. Either $\mathcal{W} \cap I\text{–}cof \subseteq J\text{–}cof$ *or* $\mathcal{W} \cap J\text{–}inj \subseteq I\text{–}inj$.

There are some advantages to know that a model category is cofibrantly generated. For instance, the next lemma provides necessary and sufficient conditions for Quillen adjunctions in case of the cofibrantly generated model.

Lemma 1.37. *Suppose* $(F, G, \phi) : \mathcal{M} \longrightarrow \mathcal{M}'$ *is an adjunction between model categories and* $\mathcal{M}$ *is cofibrantly generated, with generating cofibrations* I *and generating trivial cofibrations* J*. Then* (F, G, ϕ) *is a Quillen adjunction if and only if* $F(f)$ *is a cofibration for* $f \in I$ *and* $F(f)$ *is a trivial cofibration for all* $f \in J$*.*

2 Differential Graded Modules

Differential graded algebras (DGAs) and differential graded modules over a DGA (DG modules) arise in different branches of mathematics particularly in algebra and topology. For example, singular chain and cochain algebras of topological spaces, Koszul complexes, and cohomology rings of topological spaces all are DGAs.

This chapter is an introduction to the category of differential graded modules. The number of definitions and results can be found in [14] and an intensive review has been given in [26]. Considering the category of DG-modules as a triangulated category Pauksztello has provided an accessible summary of [28] in his thesis [30]. However we will mainly follow the exposition given in [4] and [12].

2.1 Basic Definitions

Definition 2.1. Let K be a commutative ring

i. A complex of $K-$modules is a sequence of $K-$linear maps

$$M : \cdots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \cdots$$

such that $\partial_i \partial_{i+1} = 0$ for all $i \in \mathbb{Z}$. The morphism ∂_i^M is called ith boundary map.

ii. A graded $K-$module is a complex in which all boundary maps are equal to 0. Thus, every complex M has an underlying graded module, denoted by $M^\natural$, and therefore the complex can be described as a pair $(M^\natural, \partial^M)$.

iii. A chain map is a homomorphism $\beta : M \longrightarrow N$ of complexes such that

$$\partial^N \circ \beta = (-1)^{|\beta|} \beta \circ \partial^M$$

where $|\beta|$ is the degree of β.

iv. A morphism of complexes is a chain map of degree 0. The set of all morphisms from M to N is denoted by $Mor_K(M,N)$.

v. Given a complex M, for each $s \in \mathbb{Z}$ we define $\Sigma^s M$ by

$$\begin{aligned}(\Sigma^s M)_i &= M_{i-s}\\ \partial_i^{\Sigma^s M} &= (-1)^s \partial_{i-s}^M\end{aligned}$$

and call it sth shift or suspension of M.

The category of complexes of $K-$modules $\mathcal{DGM}(K)$, is a category whose objects are complexes of $K-$modules and morphisms are the morphisms between complexes. Furthermore the category of graded $K-$modules is denoted by $\mathcal{GM}(K)$.

Example 2.2. Suppose M and N are complexes of $K-$modules.

i. $Hom_K(M,N)$ is a complex of $K-$modules with

$$(Hom_K(M,N))_d = \prod_{i\in\mathbb{Z}} Hom_K(M_i, N_{i+d})$$

and the boundary map $\partial^{Hom_K(M,N)}$ acts on $\beta \in Hom_K(M,N)$ by

$$\partial^{Hom_K(M,N)}(\beta) = \partial^N \circ \beta - (-1)^{|\beta|}\beta \circ \partial^M.$$

ii. The tensor product $N \otimes_K M$ is a complex of $K-$modules with

$$(N \otimes_K M)_d = \coprod_{i+j=d} N_i \otimes_K M_j$$

and the boundary map $\partial^{N\otimes_K M}$ acts on $m \otimes n$ by

$$\partial^{N\otimes_K M}(n\otimes m) = \partial^N(n)\otimes m + (-1)^{|n|} n \otimes \partial^M(m).$$

Definition 2.3. A differential graded algebra R (DG algebra) is a pair, $(R^\natural, \partial^R)$ consisting of a graded $K-$module $R^\natural$ and a boundary map ∂^R satisfying the Leibniz rule

$$\partial^R(rr') = \partial^R(r)r' + (-1)^{|r|} r\partial^R(r')$$

for all r and r' in R. A DG algebra R is commutative if

$$rr' = (-1)^{|r||r|'} r'r \qquad \forall r, r' \in R.$$

A morphism of DG algebras $\phi : R \longrightarrow S$ is a morphism of their underlying complexes $R^\natural$ and $S^\natural$ such that $\phi(rr') = \phi(r)\phi(r')$ and $\phi(1_R) = 1_S$.

Definition 2.4. A differential graded module (DG module) M over a DG algebra R is a pair $(M^\natural, \partial^M)$ where $M^\natural$ is a $R^\natural$ module and the boundary map ∂^M satisfies the Leibniz rule

$$\partial^M(rm) = \partial^R(r)m + (-1)^{|r|} r\partial^M(m) \qquad \forall r \in R, \forall m \in M$$

If M and N are DG $R-$modules, then a map $\beta : M \longrightarrow N$ is a morphism of DG modules if it is a morphism of underlying complexes and

$$\beta(rm) = r\beta(m) \qquad \forall r \in R, \forall m \in M.$$

Additionally, $\mathcal{DGM}(R)$ denotes the category of DG $R-$modules. If M and N are DG modules, then $Mor_R(M, N)$ denotes the set of morphisms from M to N.

Example 2.5. Consider a family $(N^u)_{u\in U}$ of DG $R-$modules. The direct product $\prod_{u\in U} N^u$ is a DG module with ith component equal to $\prod_{u\in U}(N^u_i)$, with the action of R and the differential given by

$$r((n^u)_{u\in U}) = (rn^u)_{u\in U} \qquad \partial((n^u)_{u\in U}) = (\partial(n^u))_{u\in U}.$$

For each $v \in U$, the canonical projection

$$\pi^v : \prod_{u\in U} N^u \longrightarrow N^v$$
$$\pi^v((n^u)_{u\in U}) = n^v$$

is a morphism of DG $R-$modules.

Example 2.6. Consider a family $(M^u)_{u\in U}$ of DG $R-$modules. The direct sum $\coprod_{u\in U} M^u$ (sometimes denoted by $\bigoplus_{u\in U} M^u$) is a subset of $\prod_{u\in U} M^u$, consisting of those $(m^u)_{u\in U}$ with $m^u = 0$ for all but finitely many $u \in U$. It is a DG submodule of the direct product, and for each $v \in U$, the canonical injection

$$i^v : M^v \longrightarrow \coprod_{u\in U} M^u$$

$$(i^v(m^v))^u = \begin{cases} m^u & \text{if } v = u, \\ 0 & \text{if } v \neq u. \end{cases}$$

is a morphism of DG $R-$modules.

2.2 Homology and Homotopy

Homology When M is a chain complex, the graded $K-$modules

$$Z(M) = \{m \in M | \partial(M) = 0\} = Ker\partial$$
$$B(M) = \{\partial(m) \in M | m \in M\} = Im\partial$$
$$C(M) = M/B(M)$$

are known respectively as the module of cycles, the module of boundaries and the module of coboundaries of M. Note that $Z(R)$ is a graded subalgebra of $R^\natural$ because if $r, r' \in Z(M)$ then by Leibnitz rule $\partial(rr') = 0$ and also $\partial(1) = 0$. Furthermore, $Z(M)$ is a graded $Z(R)-$submodule of $M^\natural$.
The relation $\partial^2 = 0$ means that $B(M) \subseteq Z(M)$, and the graded $K-$module

$$H(M) = Z(M)/B(M)$$

is called the *homology* of M. The ith component of $H(M)$ is denoted by $H_i(M)$ rather than $(H(M))_i$ and we apply similar convention for $Z(M)$ and $B(M)$. The image of $z \in Z(M)$ in $H(M)$ is shown by $cls(z)$ or $[z]$ and is called the homology class of z.

The proof of following statement is quite straightforward.

Proposition 2.7. *The homology defines a functor*

$$H : \mathcal{DGM}(R) \longrightarrow \mathcal{GM}(H(R))$$

which commutes with the suspension.

Definition 2.8. A morphism $\beta : M \longrightarrow N$ in $\mathcal{DGM}(R)$ is a *quasi-isomorphism* if $H(\beta)$ is an isomorphism. The symbol $\simeq$ indicates quasi-isomorphisms.

Proposition 2.9. *If $\beta : M \longrightarrow N$ is a quasi-isomorphism of complexes of $K-$modules, then the following hold.*

i. β is surjective if and only if $Z(\beta)$ is surjective.

ii. β is injective if and only if $B(\beta)$ is injective.

Proof. We prove just the first statement but the second one can be proved similarly. First of all a direct computation shows that the sequences

$$0 \longrightarrow B(M) \longrightarrow Z(M) \longrightarrow H(M) \longrightarrow 0$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(M) & \longrightarrow & M & \longrightarrow & \Sigma B(M) & \longrightarrow & 0 \\ & & & & m & \longmapsto & \partial^M(m) & & \end{array}$$

are exact. Consider the two commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & B(M) \arrow[r] \arrow[d, "B(\beta)"] & Z(M) \arrow[r] \arrow[d, "Z(\beta)"] & H(M) \arrow[r] \arrow[d, "H(\beta)"] & 0 \\ 0 \arrow[r] & B(N) \arrow[r] & Z(N) \arrow[r] & H(N) \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z(M) \arrow[r] \arrow[d, "Z(\beta)"] & M \arrow[r] \arrow[d, "\beta"] & \Sigma B(M) \arrow[r] \arrow[d, "\Sigma B(\beta)"] & 0 \\ 0 \arrow[r] & Z(N) \arrow[r] & N \arrow[r] & \Sigma B(N) \arrow[r] & 0 \end{tikzcd}$$

The Snake lemma [36, 1.3.2] says that if β is surjective, then the lower diagram implies $B(\beta)$ is surjective, and then the upper diagram shows that $Z(\beta)$ is surjective. Conversely, if $Z(B)$ is surjective then the upper diagram implies that $B(\beta)$ is surjective. The surjectivity of both $Z(\beta)$ and $B(\beta)$ in the lower diagram shows that β is surjective. $\square$

The next theorem shows how the homology functor behaves on an exact sequence. In fact, this is the most important property of the homology functor. For a full description of connecting homomorphism and following theorem see [35, 4.2].

Theorem 2.10. *For every exact sequence of DG $R-$modules*

$$\boldsymbol{E} : 0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

the connecting homomorphism $\partial^{\boldsymbol{E}}$ appears in an exact sequence

$$H(L) \xrightarrow{H(\alpha)} H(M) \xrightarrow{H(\beta)} H(N) \xrightarrow{\partial^{\boldsymbol{E}}} \Sigma H(L) \xrightarrow{\Sigma H(\alpha)} \Sigma H(M)$$

of morphisms of graded $H(R)-$modules.

Homotopy

Definition 2.11. Let M and N be DG $R-$modules.

i. A homomorphism of DG $R-$modules $\alpha : M \longrightarrow N$ is said to be null homotopic if there exists a homomorphism of $R-$modules $\xi : M \longrightarrow N$, such that

$$\alpha = \partial^N \circ \xi + (-1)^{|\alpha|} \xi \circ \partial^M$$

A map ξ as above is called null homotopy for α, if $|\xi| = |\alpha| + 1$.

ii. Let $\beta, \beta' : M \longrightarrow N$ be homomorphisms of DG $R-$modules. β is said to be homotopic to β', denoted by $\beta \sim \beta'$, if $\beta' - \beta$ is null homotopic, that is, if

$$\beta' = \beta + \partial^N \circ \xi + (-1)^{|\beta|} \xi \circ \partial^M$$

for some $R-$linear map $\xi : M \longrightarrow N$.

The next statement provides useful ways to factor general morphisms of DG $R-$modules.

Proposition 2.12. *[4, 6.2.7] Let $\beta : M \longrightarrow N$ be a morphism of DG $R-$modules.*

i. There exists a diagram of DG modules

$$\begin{tikzcd}
 & M' \arrow[d, two heads, "\beta'"] \arrow[dr, "\pi^M"] & \\
M \arrow[ur, hook, "\iota^M"] \arrow[r, "\beta"'] & N & M \arrow[l, "\beta"]
\end{tikzcd}$$

with β' *surjective,* $\beta'\iota^M = \beta$, $\beta\pi^M \sim \beta'$, $\pi^M\iota^M = 1_M$, *and* $\iota^M\pi^M \sim 1_{M'}$.

ii. There exists a diagram of DG modules

$$\begin{tikzcd}
N \arrow[dr, hook, "\iota^N"'] & M \arrow[l, "\beta"'] \arrow[r, "\beta"] \arrow[d, hook, "\beta'"] & N \\
 & N' \arrow[ur, "\pi^N"'] &
\end{tikzcd}$$

with β' *injective,*$\beta' \sim \iota^N\beta$, $\pi^N\beta' = \beta$, $\pi^N\iota^N = 1_N$, *and* $\iota^N\pi^N \sim 1_{N'}$.

In either case, β' *is a quasi-isomorphism if and only if* β *is one.*

2.3 Functors

Definition 2.13. Suppose M and N are DG $R-$modules. A homomorphism of DG modules over R or an $R-$linear map is a morphism $\beta : M \longrightarrow N$ of the underlying complexes of $K-$modules, such that

$$\beta(rm) = (-1)^{|\beta||r|} r\beta(m) \quad \forall r \in R, \forall m \in M$$

The set of all homomorphisms is denoted by $Hom_R(M, N)$.

Remark 2.14. $Hom_R(M, N)$ is a DG $K-$module and the boundary map $\partial^{Hom_R(M,N)}$ acts on $\beta \in Hom_R(M, N)$ by

$$\partial^{Hom_R(M,N)}(\beta) = \partial^N \circ \beta - (-1)^{|\beta|}\beta \circ \partial^M.$$

Therefore the set of the cycles of degree zero is set of all degree zero homomorphisms β such that $\partial^N \circ \beta{=}\beta \circ \partial^M$ in other words

$$Z_0(Hom_R(M, N)) = Mor_R(M, N).$$

Moreover, looking at a homomorphism $\beta : M \longrightarrow N$ as an element of the complex $Hom_R(M,N)$, it is a chain map if and only if it is a cycle, and is null homotopic if and only if it is a boundary in $Hom_R(M,N)$ and therefore

$$H_i(Hom_R(M,N)) = Z_i(Hom_R(M,N))/\sim$$

for each $i \in \mathbb{Z}$.

Proposition 2.15. *[4, 3.1.4] For any DG $R-$module M and N the set $Hom_R(M,N)$ is a subcomplex of $Hom_K(M,N)$. Moreover the map*

$$Hom_R(-,-) : \mathcal{DGM}(R)^{op} \times \mathcal{DGM}(R) \longrightarrow \mathcal{DGM}(K)$$

is a functor which commutes with the forgetful functor

$$Hom_R(M,N)^\natural = Hom_{R^\natural}(M^\natural, N^\natural)$$

Remark 2.16. The functors $Hom_R(M,-)$ and $Hom_R(-,M)$ are respectively left and right exact. In addition, the property of being linearly split is preserved by these functors.

Lemma 2.17. *If M and N are DG $R-$modules then, the following hold.*

i. $Hom_R(M, \Sigma^s N) = \Sigma^s Hom_R(M,N)$

ii. $Hom_R(\Sigma^{-s} M, N) \cong \Sigma^s Hom_R(M,N)$

iii. $Mor_R(M, \Sigma^{-s} N) = Z_s Hom_R(M,N) \cong Mor_R(\Sigma^s M, N)$

Proof. (i) For a given integer s, suppose σ^s is a natural transformation of the identity functor into Σ^s. The composition of chain maps of complexes of $K-$modules

$$\Sigma^s Hom_R(M,N) \xrightarrow{\sigma^{-s}} Hom_R(M,N) \xrightarrow{Hom_R(M,\sigma^s)} Hom_R(M,\Sigma^s N)$$

has degree 0 and therefore it is a morphism of complexes. In addition, for each $n \in \mathbb{Z}$ the following equalities of $K-$modules exists.

$$(\Sigma^s Hom_R(M,N))_n = \prod_{i\in\mathbb{Z}} Hom_K(M_i, N_{i+n-s}) = Hom_R(M, \Sigma^s N)_n.$$

Hence, the composition is the identity morphism in each degree and therefore it is the identity map of complexes of $K-$modules in the first statement.

(ii) Consider the composition of the chain maps

$$Hom_R(\Sigma^{-s}M,N) \xrightarrow{Hom_R(\sigma^{-s},N)} Hom_R(M,N) \xrightarrow{\sigma^s} \Sigma^s Hom_R(M,N)$$

which leads to a natural morphism of complexes of $K-$modules

$$\beta \longmapsto (-1)^{s|\beta|}\sigma^s(\beta \circ \sigma^{-s})$$

that is in fact an isomorphism.

(iii) Finally, the first and the second statements yield the last statement. □

Lemma 2.18. *For a DG module M and a family $(N^u)_{u\in U}$ of DG $R-$modules,*

$$Hom_R(M, \prod_{u\in U} N^u) \cong \prod_{u\in U} Hom_R(M, N^u) \tag{2.3.1}$$

$$Mor_R(M, \prod_{u\in U} N^u) \cong \prod_{u\in U} Mor_R(M, N^u) \tag{2.3.2}$$

Furthermore, for a family $(M^u)_{u\in U}$ of DG $R-$modules and a DG $R-$module N,

$$Hom_R(\coprod_{u\in U} M^u, N) \cong \prod_{u\in U} Hom_R(M^u, N) \tag{2.3.3}$$

$$Mor_R(\coprod_{u\in U} M^u, N) \cong \prod_{u\in U} Mor_R(M^u, N) \tag{2.3.4}$$

Proof. The maps $\beta \longmapsto (\pi^u\beta)_{u\in U}$ and $(l \mapsto (\beta^u(l))_{u\in U}) \longleftarrow\!\!| (\beta^u)_{u\in U}$ define the desired isomorphism for 2.3.1 and 2.3.2. In addition, the maps $\mu \longmapsto (\mu i^u)_{u\in U}$ and $((m^u)_{u\in U} \mapsto \sum_{u\in U} \mu^u(m^u)) \longleftarrow\!\!| (\mu^u)_{u\in U}$ define the desired isomorphism for 2.3.3 and 2.3.4. □

Definition 2.19. Let M be a DG $R-$module and L be a DG R^o-module. The actions of R^o and R on L and M define a morphism

$$\delta^{RLM} : R \otimes_K L \otimes_K M \longrightarrow L \otimes_K M$$

$$r \otimes l \otimes m \longmapsto (-1)^{|r||l|}((lr) \otimes m - l \otimes (rm))$$

of complexes of $K-$modules. In this case we define, the tensor product of DG modules by

$$L \otimes_R M = Coker\, \delta^{RLM}.$$

Proposition 2.20. *[4, 3.2.4] For any DG $R-$module M and DG R^o-module L, $L \otimes_R M$ is a quotient of $R \otimes_K L \otimes_K M$. Moreover the map*

$$(- \otimes_R -) : \mathcal{DGM}(R^o) \times \mathcal{DGM}(R) \longrightarrow \mathcal{DGM}(K)$$

is a functor which commutes with the forgetful functor

$$(L \otimes_R M)^\natural = L^\natural \otimes_{R^\natural} M^\natural$$

Remark 2.21. The tensor product of DG modules commutes with the suspension and the coproduct, i.e,

$$(\Sigma^s L) \otimes_R M = \Sigma^s (L \otimes_R M)$$
$$L \otimes_R \Sigma^s M \cong \Sigma^s (L \otimes_R M)$$
$$L \otimes_R (\coprod_{u \in U} M^u) \cong \coprod_{u \in U} (L \otimes_R M^u)$$

Several fundamental isomorphisms of DG bimodules are just restated in the rest of this section. However more details can be found in [4, 3.3]. For the rest of this section, suppose Q, R, S and T are DG algebras.

Proposition 2.22. *Let R be a DG bimodule over R and R^o, L be a DG bimodule over Q and R^o, M be a DG bimodule over R and S. The evaluation morphisms*

$$\begin{array}{rcl} Hom_{R^o}(R, L) & \longrightarrow & L \\ \alpha & \longmapsto & \alpha(1) \end{array} \qquad \begin{array}{rcl} Hom_R(R, M) & \longrightarrow & M \\ \beta & \longmapsto & \beta(1) \\ (r \mapsto (-1)^{|m||r|} rm) & \longleftarrow & m \end{array}$$

are isomorphisms of DG bimodules over Q and R^o, and R and S, respectively.

Proposition 2.23. *Let L be a DG bimodule over Q and R, M be a DG bimodule over S and T, and N be a DG bimodule over R and S. The swap morphism*

$$\begin{array}{rcl} Hom_R(L, Hom_S(M, N)) & \longrightarrow & Hom_S(M, Hom_R(L, N)) \\ \alpha & \longmapsto & (m \mapsto (l \mapsto (-1)^{|l||m|} \alpha(l)(m))) \\ (l \mapsto (m \mapsto (-1)^{|l||m|} \beta(m)(l))) & \longleftarrow & \beta \end{array}$$

is an isomorphism of DG bimodules over Q^o and T^o.

Proposition 2.24. *Let L be a DG bimodule over Q and R^o, M be a DG bimodule over R and S, and N be a DG bimodule over S and T. The adjointness morphism*

$$\begin{array}{rcl} Hom_S(L \otimes_R M, N) & \longrightarrow & Hom_{R^o}(L, Hom_S(M,N)) \\ \beta & \longmapsto & (l \mapsto (m \mapsto \beta(l \otimes m))) \\ (l \otimes m \mapsto \gamma(l)(m)) & \longleftarrow\!\shortmid & \gamma \end{array}$$

is an isomorphism of DG bimodules over Q^o and T. Moreover, the transposition morphism

$$\begin{array}{rcl} L \otimes_R M & \longrightarrow & M \otimes_{R^o} L \\ l \otimes m & \longleftrightarrow & (-1)^{|l||m|} m \otimes l \end{array}$$

is an isomorphism of DG bimodules over Q and S.

2.4 Constructions of DG Modules

Pullback and Pushout

Proposition 2.25. *Let $M \xrightarrow{\beta} N \xleftarrow{\gamma} N'$ be morphisms in $\mathcal{DGM}(R)$*

i. The pullback of the pair (β, γ) is the DG module

$$M \times_N N' = Ker\Big((\beta, -\gamma) : M \oplus N' \longrightarrow N \Big)$$

which appears in a commutative pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & Ker\,\beta' \arrow[r] \arrow[d, "\underline{\gamma}"'] & M \times_N N' \arrow[r, "\beta'"] \arrow[d, "\gamma'"'] & N' \arrow[d, "\gamma"'] \\ 0 \arrow[r] & Ker\,\beta \arrow[r] & M \arrow[r, "\beta"'] & N \end{tikzcd}$$

where $\gamma'(m, n') = m$, $\beta'(m, n') = n'$, and $\underline{\gamma}$ is the restriction of γ'.

ii. $\underline{\gamma}$ is an isomorphism.

iii. If β is surjective, then so is β'.

iv. If β is surjective-quasi-isomorphism, then so is β'.

Proof. The first and second statements can be verified by straightforward computations. If β is a surjective quasi-isomorphism, then the homology exact sequence of the bottom row shows that $H(Ker\ \beta) = 0$. Hence, $H(Ker\ \beta') = 0$; showing $H(\beta')$ is a bijection by using the properties of homology long exact sequence. $\square$

A dual argument of 2.25, which results in the next proposition, can be conducted.

Proposition 2.26. *Let* $M' \xleftarrow{\alpha} M \xrightarrow{\beta} N$ *be morphisms in* $\mathcal{DGM}(R)$

i. *The pushout of the pair* (α, β) *is the DG module*

$$M' \oplus^M N = Coker\Big(\left(\begin{smallmatrix}-\alpha \\ \beta\end{smallmatrix}\right) : M \longrightarrow M' \oplus N \Big)$$

Setting $L = Im\left(\begin{smallmatrix}-\alpha \\ \beta\end{smallmatrix}\right)$, *then we have the following pushout commutative diagram*

$$\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "\alpha"] & N \arrow[r] \arrow[d, "\alpha'"] & Coker(\beta) \arrow[r] \arrow[d, "\bar{\alpha}"] & 0 \\
M' \arrow[r, "\beta'"] & M' \oplus^M N \arrow[r] & Coker\beta' \arrow[r] & 0
\end{tikzcd}$$

where $\beta(m') = (m', 0) + L, \alpha'(n) = (0, n) + L, \bar{\alpha}(n + Im(\beta)) = \alpha'(n) + Im(\beta')$.

ii. $\bar{\alpha}$ *is an isomorphism.*

iii. *If* β *is injective,* β' *is injective.*

iv. *If* β *in injective quasi-isomorphism, so is* β'.

Limits and colimits At this stage, limits and colimits will be constructed in $\mathcal{DGM}(R)$ and several results about them and their relations with homology will be provided. Let U be a partially ordered set, and let

$$\nabla(U) = \{(u, v) \in U \times U | u \leq v\}$$

be the superdiagonal. Along with a family $(N^u)_{u \in U}$ we consider the family

$$(N^{uv} | N^{uv} = N^u)_{(u,v) \in \nabla(U)}.$$

Definition 2.27. An inverse system of morphisms in $\mathcal{DGM}(R)$ is a family $\mathbb{N} = (\nu^{uv} : N^v \to N^u)_{(u,v) \in \nabla(U)}$ such that

$$\nu^{tu}\nu^{uv} = \nu^{tv} \quad \text{for } t \leq u \leq v \qquad \text{and} \qquad \nu^{uu} = id^{N^u} \quad \text{for all } u.$$

The inverse limit, $\varprojlim \mathbb{N} = \varprojlim_u N^u$ is defined by the exactness of the sequence

$$0 \longrightarrow \varprojlim_u N^u \xrightarrow{\zeta_{\mathbb{N}}} \prod_{u\in U} N^u \xrightarrow{\theta_{\mathbb{N}}} \prod_{(u,v)\in\nabla(U)} N^{uv} \tag{2.4.1}$$

$$(n^u)_{u\in U} \overset{\theta}{\longmapsto} (n^u - \nu^{uv}(n^v))_{(u,v)\in\nabla(U)}$$

It is a DG module. For each u the composition of ζ with the canonical map π^u yields a morphism $\nu^u : \varprojlim_u N^u \longrightarrow N^u$ with $\nu^u = \nu^{uv}\nu^v$ for all $(u,v) \in \nabla(U)$.

If $\mathbb{M} = (\mu^{uv} : M^v \to M^u)_{(u,v)\in\nabla(U)}$ is an inverse system over U, then a morphism $\beta : \mathbb{M} \longrightarrow \mathbb{N}$ is a family $(\beta^u : M^u \longrightarrow N^u)_{u\in U}$ of morphisms of DG $R-$modules such that $\beta^u\mu^{uv} = \nu^{uv}\beta^v$ for all $(u,v) \in \nabla(U)$. The map

$$\varprojlim \beta : \varprojlim \mathbb{M} \longrightarrow \varprojlim \mathbb{N}$$

$$(m^u)_{u\in U} \longmapsto (\beta^u(m^u))_{u\in U}$$

is a morphism in $\mathcal{DGM}(R)$, called the limit of β.

Remark 2.28. A sequence $\mathbb{L} \xrightarrow{\alpha} \mathbb{M} \xrightarrow{\beta} \mathbb{N}$ of morphisms of inverse systems is exact if each sequence of DG $R-$modules $L^u \xrightarrow{\alpha^u} M^u \xrightarrow{\beta^u} N^u$ is exact. Constructing inverse limits of a sequence by using (2.4.1) and considering the Snake Lemma shows that limit is left exact which means that an exact sequence of inverse systems $0 \longrightarrow \mathbb{L} \xrightarrow{\alpha} \mathbb{M} \xrightarrow{\beta} \mathbb{N}$ induces an exact sequence of DG $R-$modules

$$0 \longrightarrow \varprojlim \mathbb{L} \xrightarrow{\varprojlim \alpha} \varprojlim \mathbb{M} \xrightarrow{\varprojlim \beta} \varprojlim \mathbb{N}$$

Proposition 2.29. *For every DG $R-$module M the functors $Hom_R(M,-)$ and $Mor_R(M,-)$ commute with limits. In other words, if $\mathbb{N}$ is an inverse system*

$$Hom_R(M, \varprojlim \mathbb{N}) \cong \varprojlim Hom_R(M, \mathbb{N}) \tag{2.4.2}$$

$$Mor_R(M, \varprojlim \mathbb{N}) \cong \varprojlim Mor_R(M, \mathbb{N}) \tag{2.4.3}$$

Proof. Notice that $Hom_R(M, \mathbb{N})$ is an inverse system of complexes of $K-$modules. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(M, \varprojlim \mathbb{N}) \arrow[r] & Hom_R(M, \prod_u N^u) \arrow[r, "\Theta'"] \arrow[d, "\cong"] & Hom_R(M, \prod_{u\le v} N^{uv}) \arrow[d, "\cong"] \\
0 \arrow[r] & \varprojlim Hom_R(M, \mathbb{N}) \arrow[r] & \prod_u Hom_R(M, N^u) \arrow[r, "\Theta''"'] & \prod_{u\le v} Hom_R(M, N^{uv})
\end{tikzcd}$$

in which $\Theta' = \Theta_{Hom_R(M,\mathbb{N})}$, $\Theta'' = Hom_R(M, \Theta_{\mathbb{N}})$ and rows are exact. As the vertical maps are isomorphisms the five lemma proves (2.4.2). Hence the induced isomorphism of groups of cycles of degree zero yields (2.4.3).

$\square$

Definition 2.30. Along with a family $(M^u)_{u\in U}$ consider the family

$$(M^{uv}|M^{uv} = M^u)_{(u,v)\in\nabla(U)}$$

A direct system of morphisms in $\mathcal{DGM}(R)$ is a family
$\mathbb{M} = (\mu^{vu} : M^u \to M^v)_{(u,v)\in\nabla(U)}$ such that

$$\mu^{vu}\mu^{ut} = \mu^{vt} \quad \text{for } t \le u \le v \qquad \text{and} \qquad \mu^{uu} = id^{M^u} \quad \text{for all } u$$

The direct limit, $\varinjlim \mathbb{M} = \varinjlim_u M^u$, is defined by the exactness of the sequence

$$\begin{array}{ccccc} \coprod_{(u,v)\in\nabla(U)} M^{uv} & \xrightarrow{\quad\gamma\quad} & \coprod_{u\in U} M^u & \xrightarrow{\quad\epsilon\quad} & \varinjlim_u M^u \longrightarrow 0 \end{array}$$

$$\sum_{(u,v)\in\nabla(U)} i^{uv}(m^{uv}) \overset{\gamma}{\longmapsto} \sum_{(u,v)\in\nabla(U)} (i^u(m^{uv}) - i^v\mu^{uv}(m^{uv}))$$

It is a DG module. For each u the composition of the map i^u with ϵ is the morphism $\mu^u : M^u \longrightarrow \varinjlim_u M^u$ with $\mu^v = \mu^w\mu^{wv}$ for all $(u, w) \in \nabla(U)$.
If $\mathbb{N} = (\nu^{uv} : N^u \to N^v)_{(u,v)\in\nabla(U)}$ is a direct system over U, then a morphism $\beta : \mathbb{M} \longrightarrow \mathbb{N}$ is a family $(\beta^u : M^u \longrightarrow N^u)_{u\in U}$ of morphisms of DG $R-$modules such that $\beta^v\nu^{vu} = \mu^{vu}\beta^u$ for all $u \le v$. The map

$$\varinjlim \beta : \varinjlim \mathbb{M} \longrightarrow \varinjlim \mathbb{N}$$

$$\epsilon_{\mathbb{M}}(\sum_{u\in U} i^u(m^u)) \longmapsto \sum_{u\in U} \mu^u\beta^u(m^u)$$

is a morphism in $\mathcal{DGM}(R)$, called the direct limit of β.

Remark 2.31. A sequence $\mathbb{L} \xrightarrow{\alpha} \mathbb{M} \xrightarrow{\beta} \mathbb{N}$ of morphisms of direct systems is exact if each sequence of DG $R-$module $L^u \xrightarrow{\alpha^u} M^u \xrightarrow{\beta^u} N^u$ is exact. Direct limits are right exact which means that an exact sequence of direct systems $\mathbb{L} \xrightarrow{\alpha} \mathbb{M} \xrightarrow{\beta} \mathbb{N} \longrightarrow 0$ induces an exact sequence of DG $R-$module

$$\varinjlim \mathbb{L} \xrightarrow{\varinjlim \alpha} \varinjlim \mathbb{M} \xrightarrow{\varinjlim \beta} \varinjlim \mathbb{N} \longrightarrow 0$$

By conducting a dual argument for 2.29 with some extra modifications, the following proposition is obtained.

Proposition 2.32. *[4, 4.6.3.1] For every DG $R-$module N the functors $Hom_R(-,N)$ and $Mor_R(-.N)$ change direct limit to inverse limit. In other words, if $\mathbb{M}$ is a direct system*

$$Hom_R(\varinjlim \mathbb{M}, N) \cong \varprojlim Hom_R(\mathbb{M}, N)$$
$$Mor_R(\varinjlim \mathbb{M}, N) \cong \varprojlim Mor_R(\mathbb{M}, N)$$

If the set U has some property, the direct limit behaves more nicely. We start analyzing these behaviors by the next definition.

Definition 2.33. An ordered set U is said to be filtered if for each pair of elements $u, v \in U$ there exists a $w \in U$ such that $u \leq w$ and $v \leq w$.

Lemma 2.34. *Let U be a filtered ordered set.*

i. For each $m \in \varinjlim \mathbb{M}$ there exists $m^u \in M^u$ such that $m = \mu^u(m^u)$.

ii. For $m^u \in M^u$ if $\mu^u(m^u) = 0$ then $\mu^{wu}(m^u) = 0$ for some $w \in U$ with $u \leq w$.

iii. For $m^u \in M^u$ and $m^v \in M^v$ if $\mu^u(m^u) = \mu^v(m^v)$ then $\mu^{wu}(m^u) = \mu^{wv}(m^v)$ for some $w \in U$ with $u, v \leq w$.

Proof. (i) By construction, m can be written in the form $m = \sum_{t \in U} \mu^t(m^t)$. Choosing $u \in U$ such that $u \geq t$ for all the ts, we get

$$m = \sum_{t \in U} \mu^u \mu^{ut}(m^t) = \mu^u(\sum_{t \in U} \mu^{ut}(m^t)).$$

(ii) If $\mu^u(m^u) = 0$, then $i^u(m^u) = \sum_{(t,v) \in \nabla(U)} (i^t(m^{tv}) - i^v \mu^{vt}(m^{tv}))$. Choose $w \in U$ with $w \geq u$ and $w \geq v$ for all vs appearing in the sum, and set

$$\gamma^w : \textstyle\coprod_{t \in U} M^t \longrightarrow M^w \qquad\qquad \gamma^w(\sum_{t \in U} i^t(m^t)) = \sum_{t \leq w} \mu^{wt}(m^t)$$

Applying the morphism γ^w to the expression for $i^u(m^u)$ we get

$$\mu^{wu}(m^u) = \sum_{(t,v)\in\nabla(U)} (\mu^{wt}(m^{tv}) - \mu^{wv}\mu^{vt}(m^{tv})) = 0.$$

(iii) Choose $t \geq u, v$. As $\mu^t(\mu^{tu}(m^u) - \mu^{tv}(m^v)) = 0$ by (ii) there is a $w \geq t$ with $\mu^{wt}(\mu^{tu}(m^u) - \mu^{tv}(m^v))$, which is $\mu^{wu}(m^u) - \mu^{wv}(m^v) = 0$. □

Proposition 2.35. *If* $0 \longrightarrow \mathbb{L} \xrightarrow{\alpha} \mathbb{M} \xrightarrow{\beta} \mathbb{N} \longrightarrow 0$ *is an exact sequence of direct systems and the set* U *is filtered, then the induced sequence*

$$0 \longrightarrow \varinjlim \mathbb{L} \xrightarrow{\varinjlim \alpha} \varinjlim \mathbb{M} \xrightarrow{\varinjlim \beta} \varinjlim \mathbb{N} \longrightarrow 0$$

of DG $R-$*modules is exact.*

Proof. It suffices to prove that $\bar{\alpha} = \varinjlim \alpha$ is injective. For $l \in Ker\ \bar{\alpha}$ choose $u \in U$ and $l^u \in L^u$ such that $l = \lambda^u(l^u)$ where λs belong to the direct system $\mathbb{L}$. Then $\mu^u\alpha^u(l^u) = \bar{\alpha}\lambda^u(l^u) = \bar{\alpha}(l) = 0$ where μs belong to the direct system $\mathbb{M}$. By using Lemma 2.34 we can find $w \geq u$ with $0 = \mu^{wu}\alpha^u(l^u) = \alpha^w\lambda^{wu}(l^u)$. The injectivity of α^w implies that $\lambda^{wu}(l^u) = 0$, and hence $l = \lambda^w\lambda^{wu}(l^u) = 0$. □

Every direct system $\mathbb{M}$ of DG modules defines a direct system

$$H(\mathbb{M}) = (H(\mu^{vu}) : H(M^u) \longrightarrow H(M^v))_{(u,v)\in\nabla(U)}$$

of morphisms of graded $H(R)-$modules. The maps $H(\mu^{vu}) : H(M^u) \longrightarrow H(\varinjlim \mathbb{M})$ induce a canonical morphism of graded $H(R)-$modules $\varinjlim H(\mathbb{M}) \longrightarrow H(\varinjlim \mathbb{M})$. The next theorem often is summarized as: Filtered colimits commute with homology.

Theorem 2.36. *If the ordered set* U *is filtered, then the canonical morphism*

$$\varinjlim H(\mathbb{M}) \longrightarrow H(\varinjlim \mathbb{M})$$

is bijective.

Proof. The direct system $\mathbb{M}$ induces direct system of complexes of $K-$modules $B(\mathbb{M})$ and $Z(\mathbb{M})$. The four direct systems are linked by exact sequences

$$0 \longrightarrow Z(\mathbb{M}) \xrightarrow{\zeta} \mathbb{M} \xrightarrow{\omega} \Sigma B(\mathbb{M}) \longrightarrow 0$$

$$0 \longrightarrow B(\mathbb{M}) \xrightarrow{\beta} Z(\mathbb{M}) \xrightarrow{\pi} H(\mathbb{M}) \longrightarrow 0$$

Proposition 2.35 results in exact sequences of complexes

$$0 \longrightarrow \varinjlim Z(\mathbb{M}) \xrightarrow{\zeta} \varinjlim \mathbb{M} \xrightarrow{\omega} \Sigma \varinjlim B(\mathbb{M}) \longrightarrow 0$$

$$0 \longrightarrow \varinjlim B(\mathbb{M}) \xrightarrow{\beta} \varinjlim Z(\mathbb{M}) \xrightarrow{\pi} \varinjlim H(\mathbb{M}) \longrightarrow 0$$

Define a map $\chi : \varinjlim H(\mathbb{M}) \longrightarrow H(\varinjlim \mathbb{M})$ as follow; for $h \in \varinjlim H(\mathbb{M})$ choose $z \in \varinjlim Z(\mathbb{M})$ with $\pi(z) = h$ and set $\chi(h) = cls(\zeta(z))$. Using the exact sequences above one checks that χ is bijective. We must show that χ is the desired canonical map induced by the family $H(\mu^u)$. Choose $u \in U$ and $z^u \in Z(M^u)$ such that $z = \mu^u(z^u) \in \varinjlim Z(\mathbb{M})$ we have $\zeta(z) = \mu^u(z^u) \in \varinjlim \mathbb{M}$. and hence $\chi(h) = cls(\zeta(z)) = cls(\mu^u(z^u)) = H(\mu^u)cls(z^u)$ which shows χ is the desired isomorphism. $\square$

Mapping Cone Another important construction of DG modules is the mapping cone which plays a fundamental role in analyzing morphisms of DG modules, by providing tools to recast properties of morphisms as properties of DG modules. For example, a morphism is a quasi-isomorphism if its mapping cone is quasi-trivial (has trivial homology groups).

Definition 2.37. Let $\beta : M \longrightarrow N$ be a morphism in $\mathcal{DGM}(R)$. The mapping cone of β is the DG module

$$Cone\ \beta = ((\Sigma M)^{\natural} \oplus N^{\natural}, \partial^C)$$

$$\partial_i^C(m,n) = (-\partial_{i-1}^M(m), \beta_{i-1}(m) + \partial_i^N(n)) \qquad \text{for} \qquad (m,n) \in (\Sigma M)_i^{\natural} \oplus N_i^{\natural}$$

∂^C has the matrix form

$$\begin{pmatrix} \partial^{\Sigma M} & 0 \\ \Sigma\beta & \partial^N \end{pmatrix}$$

Remark 2.38. Mapping cones are natural in the sense that for every commutative diagram of DG $R-$modules

$$\begin{tikzcd} M \arrow[r, "\beta"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ M' \arrow[r, "\beta'"'] & N' \end{tikzcd}$$

the map $Cone\ \beta \longrightarrow Cone\ \beta'$ given by $(m,n) \mapsto (\mu(m), \nu(n))$ is a morphism of DG $R-$modules and it is an isomorphism if ν and μ are. In addition, the mapping cone sequence of β

$$\begin{array}{l} \mathbf{C}(\beta) : 0 \longrightarrow N \xrightarrow{i} Cone\ \beta \xrightarrow{\pi} \Sigma M \longrightarrow 0 \\ \qquad\qquad\quad n \longmapsto (0,n) \\ \qquad\qquad\qquad\qquad (m,n) \longmapsto m \end{array}$$

is a linearly split sequence of DG $R-$modules. Linearly split means split as graded $R^{\natural}-$modules.

Lemma 2.39. *A morphism* $\beta : M \longrightarrow N$ *is a quasi-isomorphism if and only if* $Cone\ \beta$ *is quasi-trivial.*

Proof. A direct computation shows the connecting homomorphism of $\mathbf{C}(\beta)$ satisfies

$$\partial^{\mathbf{C}(\beta)} = H(\Sigma\beta) = \Sigma H(\beta).$$

Thus, there is an exact sequence of morphisms of graded $H(R)-$modules

$$\Sigma^{-1} H(Cone\ \beta) \longrightarrow H(M) \xrightarrow{H(\beta)} H(N) \longrightarrow H(Cone\ \beta)$$

which shows the result immediately. $\square$

Now we describe the behavior of the cone construction under the homomorphism and tensor functors.

Proposition 2.40. *[4, 5.2.3] Let* $\beta : M \longrightarrow N$ *be a morphism in* $\mathcal{DGM}(R)$ *and let* $Cone\ \beta$ *be its mapping cone. For any DG* $R-$*module* P *and every DG module* L *over* R^o *there are natural isomorphisms of complexes*

$$Hom_R(P, Cone\ \beta) \cong Cone\ Hom_R(P, \beta) \tag{2.4.4}$$

$$Hom_R(Cone\ \beta, P) \cong Cone\ Hom_R(\Sigma\beta, P) \tag{2.4.5}$$

$$L \otimes_R (Cone\ \beta) \cong Cone\ L \otimes_R \beta \tag{2.4.6}$$

2.5 Freeness

Free objects on any graded set can be constructed in the category $\mathcal{DGM}(R)$. The results from Lemma 2.49 show such DG modules are always contractible therefore they carry no information of the differential structure of $\mathcal{DGM}(R)$. However, there exists a class of objects called semi-free modules carrying enough information on the structure of $\mathcal{DGM}(R)$ and behaving similarly to free objects on the categories of ordinary modules and CW-complexes in category of topological spaces. In this section, these two classes of objects and their properties are introduced.

Definition 2.41. Let R be DGA, L be a DG $R-$module and X be a graded set.

i. [26] A DG $R-$module is called *DG-free*, if it is isomorphic to a direct sum of suspensions of R. We denote $R^{(X)}$ as the DG-free module

$$\coprod_{x\in X} Re_x \text{ where } |e_x| = |x| \text{ and } \partial(e_x) = 0$$

In fact $R^{(X)}$ is free on a basis of cycles.

ii. A subset E of L is called a *semi-basis* if it is a basis of $L^\natural$ over $R^\natural$ and has a decomposition $E = \bigsqcup_{u\geq 0} E^u$ as a union of disjoint graded sets E^u such that

$$\partial(E^u) \subset R(\bigsqcup_{i<u} E^i) \text{ for all } u \in \mathbb{Z}$$

A DG module which has some semi-basis is said to be *semi-free*.

iii. A *semi-free filtration* of L is a sequence of DG submodules

$$\mathcal{L} = \{\cdots \subseteq L^{u-1} \subseteq L^u \subseteq L^{u+1} \subseteq \cdots\}$$

with $L = \bigcup_{u\in\mathbb{Z}} L^u$, $L^{-1} = 0$, and L^u/L^{u-1} DG free for all $u \in \mathbb{Z}$.

The next proposition gives a better explanation of semi-free modules and makes it easier to deal with them.

Proposition 2.42. *[4, 8.2,2] For a DG module L the following are equivalent.*

i. L is semi-free.

ii. L has a semi-free filtration.

iii. L has a well ordered basis E, such that for every $e \in E$

$$\partial(e) \in RE^{<e} \textit{ where } E^{<e} = \{e' \in E | e' < e\}.$$

Proof. (i)⇒ (ii). If $E = \bigsqcup_{u \geq 0} E^u$ is a semi-basis, then $L^u = R(\bigsqcup_{i \leq u} E^i)$ is a DG submodule of L, and the inclusions $L^{u-1} \subseteq L^u$ define a semi-free filtration.

(ii)⇒ (iii). If $\mathcal{L}$ is a semi-free filtration, then for each $u \geq 0$ choose first for $(L^u/L^{u-1})^\natural$ a basis of cycles, then lift this basis to a set $E^u \subseteq L^u$. Obviously, $E = \bigsqcup_{u \geq 0} E^u$ is a basis of the graded $R^\natural-$module $L^\natural$. Define each element of $E^{u'}$ to be smaller than any element of E^u if $u' \leq u$ and then well ordering each E^u by applying Zermelo's theorem of well-ordering, now the desired ordering can be imposed on E.

(iii)⇒ (i). Let E be a well ordered basis with $\partial(E) \subseteq RE^{<e}$. Set $E^{-1} = \emptyset$ and $L^{-1} = 0$. Define recursively, $E^u = \{e \in E | \partial(e) \in L^{u-1}\}$ and $L^u = RE^u$. Since L^u is a DG submodule of L, and $\{e + L^{u-1} | e \in E^u\}$ is a basis for $(L^u/L^{u-1})^\natural$ consisting of cycles, the set $E' = \bigcup_{u \geq 0} E^u$ generates a semi-free submodule of L. We claim $E' = E$. Suppose $E' \neq E$, let e be the initial element of $E \setminus E'$, by assumption, $\partial(e)$ is a linear combination of elements $e' \in E$ with $e' < e$, so $e' \in E'$ by the choice of e. Therefore, there exists $u \geq$ such that $\partial(e) \in E^u$ which shows that $e \in E^{u+1}$. This is a contradiction. □

Remark 2.43. The minimal elements of the well ordered basis E must be cycles.

The next example shows that complexes of free modules over an ordinary ring need not be semi-free DG modules.

Example 2.44. Let $R = \mathbb{Z}/(4)$ and M be the DG module

$$\ldots \longrightarrow Re_{j+1} \xrightarrow{\partial_{j+1}} Re_j \xrightarrow{\partial_j} Re_{j-1} \longrightarrow \ldots$$

where Re_j is a free $R-$module with basis $\{e_j\}$ and $\partial(e_j) = 2e_{j-1}$ for all $j \in \mathbb{Z}$. If M were semi-free then by 2.42 $M^\natural$ has a well ordered basis whose minimal elements are cycles. This is impossible, as every basis of $M^\natural$ has the form $\{\pm e_j\}_{j\in\mathbb{Z}}$ and such a set contains no cycles.

Definition 2.45. A *semi-free resolution* of an $R-$module M is a quasi-isomorphism $\epsilon : L \longrightarrow M$ from a semi-free DG module L; such a resolution is *strict* if the map ϵ is surjective.

Semi-free resolutions play a quite similar role to CW-complexes in the category of topological spaces. The next theorem, which has been proved in [14, 6.6] and Section 3.6, is analogous to the CW approximation theorem for topological spaces. Note that, we omit the word "strict" if there is no confusion.

Theorem 2.46. *Every DG module has a strict semi-free resolution.*

Categorically free DG module A definition of freeness in $\mathcal{DGM}(R)$ based on the terminology introduced in [4, 8.4.1] is provided here.

Definition 2.47. A subset Y of a DG $R-$module L is said to be *categorically free* if for each DG module M over R and every homogeneous map $\kappa : Y \longrightarrow M$ of degree 0 there exists a unique $R-$linear morphism $\tilde{\kappa} : L \longrightarrow M$ with $\tilde{\kappa}(y) = \kappa(y)$ for all $y \in Y$. The DG module L is categorically free over R if it contains a categorically free subset.

Lemma 2.48. *[4, 8.4.4] For each DG $R-$module M there exists a surjective morphism $L \longrightarrow M$, where L is a categorically free DG module.*

Lemma 2.49. *[4, 8.4.5] If L is a categorically free DG module then L is semi-free and contractible.*

2.6 Projectivity

Recall that for an ordinary ring S, an $S-$module P is projective if the functor $Hom_S(P, -)$ is exact. However the situation for DG modules is different since the category of DG modules over R can be enriched in $\mathcal{DGM}(K)$. In this section we try to investigate more properties of the category $\mathcal{DGM}(R)$ by studying varieties of projectivities defined by the $Hom(P,-)$ functor.

Through the rest of this section A and R denote a graded algebra and a DG algebra respectively.

Definition 2.50. A DG $R-$module P is said to be

- *linearly projective* if $Hom_R(P,-)$ preserves surjective morphisms.
- *homotopically projective* if $Hom_R(P,-)$ preserves quasi-isomorphisms.
- *semi-projective* if $Hom_R(P,-)$ preserves surjective quasi-isomorphisms.

These definitions are generalizations of the first definition of [3] for the category of DG modules.

Remark 2.51. For P in $\mathcal{DGM}(R)$ the following hold.

i. P is respectively, linearly projective, homotopically projective and semi-projective over R if $\Sigma^i P$ has the corresponding property for some $i \in \mathbb{Z}$, if and only if $\Sigma^i P$ has that property for all $i \in \mathbb{Z}$.

ii. If $P = \coprod_{u \in U} P^u$ then P is respectively, linearly projective, homotopically projective and semi-projective over R if and only if P^u has the corresponding property for every $u \in U$.

Definition 2.52. Let P be a graded $A-$module. P is projective if the functor

$$Hom_A(P,-) : \mathcal{GM}(A) \longrightarrow \mathcal{GM}(K)$$

preserves surjective morphisms.

The following proposition is analogous to the properties of projective modules in the ordinary sense and therefore the proof is omitted. Recall that a morphism is a homomorphism of degree 0 by 2.14.

Proposition 2.53. *For a graded $A-$module P the following are equivalent.*

i. P is projective.

ii. *$Hom_A(P,-)$ preserves exact sequences.*

iii. *If $\alpha : P \longrightarrow N$ is a homomorphism and $\beta : M \longrightarrow N$ is a surjective morphisms then there is a homomorphism $\gamma : P \longrightarrow M$ such that $\alpha = \beta\gamma$ which means the diagram below commutes.*

$$\begin{tikzcd} & M \arrow[d, "\beta"] \\ P \arrow[ur, dotted, "\gamma"] \arrow[r, "\alpha"'] & N \end{tikzcd}$$

iv. *If $\rho : M \longrightarrow P$ is a surjective morphism, then there is a morphism $\sigma : P \longrightarrow M$ such that $\rho\sigma = 1_P$.*

v. *P is a direct summand of some graded free $A-$module L.*

Proposition 2.54. *Let $(\ \iota^{u,u-1} : P^{u-1} \longrightarrow P^u\ )_{u\in\mathbb{Z}}$ be a direct system of monomorphisms of graded $A-$modules with $P^u = 0$ for $u \ll 0$. If $C^u = Coker\ \iota^{u,u-1}$ is projective for all $u \in \mathbb{Z}$, then $P = \varinjlim_u P^u$ is projective and $P \cong \coprod_{u\in\mathbb{Z}} C^u$.*

Proof. As the direct limit is left exact, the canonical morphisms $\iota^u : P^u \longrightarrow P$ are injective and $P = \bigcup_{u\in\mathbb{Z}} Im(i^u)$, so for the rest of the proof we identify P^u with its image in P. If $\rho^u : P^u \longrightarrow C^u$ is the canonical surjection for all $u \in \mathbb{Z}$ there is a morphism $\sigma^u : C^u \longrightarrow P^u$ such that $\rho^u\sigma^u = 1_{C^u}$. Set $Q = \coprod_{u\in\mathbb{Z}} C^u$, write $y \in Q$ in the form $y = (\cdots, y^u, \cdots)$ with $y^u \in C^u$, and define a map $\sigma : Q \longrightarrow P$ by

$$\sigma(y) = \sum_{u\in\mathbb{Z}} \sigma^u(y^u).$$

We claim that σ is an isomorphism which shows that P is projective by 2.51. If $y \neq 0$, then $v = Sup\{u | y^u \neq 0\}$ is finite and $\sigma(y) \in P^v$. Thus $\rho^v\sigma(y) = y^v$ which is not zero, therefore σ is injective. To show $P^u \subseteq Im(\sigma)$ for all $u \in \mathbb{Z}$ we apply induction on u. For $u \ll 0$, $P^u = 0$ so $P^u \subseteq Im(\sigma)$. Suppose $P^{v-1} \subseteq Im(\sigma)$ for some $v \in \mathbb{Z}$. For each $x \in P^v$ we have

$$\begin{aligned} \rho^v(x - \sigma^v\rho^v(x)) &= \rho^v(x) - \rho^v\sigma^v\rho^v(x) \\ &= 0 \qquad\qquad\qquad \text{as } \rho^u\sigma^u = 1_{C^u} \end{aligned}$$

hence $x - \sigma^v\rho^v(x)$ is in $Ker\ (\rho^v) = P^{v-1}$, and therefore there exists a $y' \in Q$ such that $x - \sigma^v\rho^v = \sigma(y')$. Set y such that $y^u = 0$ for $u \neq v$ and $y^v = \rho^v(x)$ so $x = \sigma(y' + y)$ which is in $Im(\sigma)$. □

Lemma 2.55. *Let Q be a DG $R-$module.*

i. Q is homotopically projective if and only if the functor $Hom_R(Q, -)$ preserves quasi-triviality.

ii. If $Q = \bigcup_{u\in\mathbb{Z}} Q^u$ for a sequence of DG submodules $Q^{u-1} \subseteq Q^u$ such that $Q^u = 0$ for $u \ll 0$, $C^u = Q^u/Q^{u-1}$ is homotopically projective for all u, and $(C^u)^\natural$ is projective over $R^\natural$ for all u, then Q is homotopically projective and $Q^\natural$ is projective over $R^\natural$.

Proof. (i) Suppose Q is a homotopically projective and E is a quasi-trivial DG module. As the map $Q \longrightarrow 0$ is a quasi-isomorphism the map $Hom_R(Q, E) \longrightarrow 0$ is also a quasi-isomorphism.

Next suppose, $Hom_R(Q, -)$ preserves quasi-triviality. If $\beta : M \longrightarrow N$ is a quasi-isomorphism then $Cone\ \beta$ is a quasi-trivial module, and therefore $Cone\ Hom_R(Q, \beta)$, which is isomorphic to $Hom_R(Q, Cone\ \beta)$, is quasi-trivial. Hence $Hom_R(Q, \beta)$ is a quasi-isomorphism.

(ii) Proposition 2.54 shows that $Q^\natural$ is projective over $R^\natural$ so by (i) it suffices to prove that if E is quasi-trivial then $H(Hom_R(Q, E)) = 0$. For each $u \in \mathbb{Z}$ the exact sequence $0 \longrightarrow Q^{u-1} \longrightarrow Q^u \longrightarrow Q^u/Q^{u-1} \longrightarrow 0$ of DG modules is linearly split, so it induces an exact sequence of complexes

$$0 \longrightarrow Hom_R(Q^u/Q^{u-1}, E) \longrightarrow Hom_R(Q^u, E) \longrightarrow Hom_R(Q^{u-1}, E) \longrightarrow 0\,.$$

Because $Q^u = 0$ for $u \ll 0$, we may assume by induction on u that $Hom_R(Q^{u-1}, E)$ is quasi-trivial. By hypothesis $H(Hom_R(Q^u/Q^{u-1}, E)) = 0$ and therefore by the long exact sequence of homology groups $H(Hom_R(Q^u, E) = 0)$. As $Q = \varinjlim_u Q^u$ we get

$$H(Hom_R(Q, E)) = H(Hom_R(\varinjlim_u Q^u, E)) \cong H(\varprojlim Hom_R(Q^u, E)) = 0$$

meaning Q is homotopically projective. □

Corollary 2.56. *A semi-free DG $R-$module L is homotopically projective, and the graded $R^\natural-$module $L^\natural$ is projective.*

Proof. If $\cdots \subseteq L^{u-1} \subseteq L^u \subseteq L^{u+1} \subseteq \cdots$ is a semi-free filtration of L, then L^u/L^{u-1} is isomorphic to a direct sum of copies of $\Sigma^s R$ for various $s \in \mathbb{Z}$. Any such module is clearly homotopically projective and therefore L^u/L^{u-1} is homotopically projective. Using Lemma 2.55 shows the results. □

Lemma 2.57. *Let Q be a semi-projective DG module, and $\alpha : Q \longrightarrow N$ be a morphism of DG modules, and let $\beta : M \longrightarrow N$ be a quasi-isomorphism of DG modules.*

i. If β is surjective, then there exists a morphism $\gamma : Q \longrightarrow M$ with $\alpha = \beta\gamma$.

ii. There exists a morphism $\gamma : Q \longrightarrow M$ such that $\alpha \sim \beta\gamma$.

Proof. (i) By hypothesis the map $Hom_R(Q, \beta)$ is a surjective quasi-isomorphism so $Z_0 Hom_R(Q, \beta) = Mor_R(Q, \beta) : Mor_R(Q, M) \longrightarrow Mor_R(Q, N)$ is a surjection by 2.9. Hence there exists a morphism $\gamma : Q \longrightarrow M$ such that $\alpha = \beta\gamma$.
(ii) By using proposition 2.12 the diagram

$$\begin{tikzcd} & M' \arrow[r, "\pi^M"] \arrow[d, "\beta'"] & M \arrow[d, "\beta"] \\ Q \arrow[ur, "\gamma'"] \arrow[r, "\alpha"'] & N \arrow[r, "1_N"'] & N \end{tikzcd}$$

can be constructed in which the square commutes up to homotopy and β' is surjective quasi-isomorphism. By the first part there exists a morphism γ' such that $\alpha = \beta\gamma'$. Thus, for $\gamma = \pi^M\gamma'$ we have $\beta\gamma = \beta\pi^M\gamma' \sim \beta'\gamma' = \alpha$.

□

Linearly projective DG module The next theorem provides a relation among different properties of a linearly projective DG module.

Theorem 2.58. *[4, 9.4.1] For a DG $R-$module P the following are equivalent.*

i. P is linearly projective.

i'. $Mor_{R^\natural}(P^\natural, -)$ preserves exact sequences.

i″. $Hom_R(P,-)$ *preserves exact sequences.*

ii. *If* $\alpha : P \longrightarrow N$ *is a chain map and* $\beta : M \longrightarrow N$ *is a surjective morphism, then there is a homomorphism* $\gamma : P^\natural \longrightarrow M^\natural$ *in* $\mathcal{GM}(R^\natural)$ *with* $\alpha = \beta\gamma$.

ii′. *If* $\alpha : P \longrightarrow N$ *is a morphism and* $\beta : M \longrightarrow N$ *is a surjective morphism, then there is a morphism* $\gamma : P^\natural \longrightarrow M^\natural$ *in* $\mathcal{GM}(R^\natural)$ *with* $\alpha = \beta\gamma$.

iii. *If* $\rho : M \longrightarrow P$ *is a surjective morphism then there exists a morphism* $\sigma : P^\natural \longrightarrow M^\natural$ *in* $\mathcal{GM}(R^\natural)$ *such that* $\rho\sigma = 1_P$.

iv. P *is a DG submodule of a free DG module* L *over* R, *with* $P^\natural$ *a direct summand of* $L^\natural$ *in* $\mathcal{GM}(R^\natural)$.

v. $P^\natural$ *is a projective graded* $R^\natural-$*module.*

Proof. $(i) \Leftrightarrow (i'')$, $(i) \Leftrightarrow (ii)$ and $(i') \Leftrightarrow (i'')$ hold by definition.
$(ii) \Leftrightarrow (ii')$ as $Z_i Hom_R(P,-) = Mor_R(p, \Sigma^{-i}(-))$ by 2.17.
$(ii') \Rightarrow (iii)$. Apply the hypothesis to $\alpha = 1_P$ and $\beta = \rho$ and set $\sigma = \gamma$.
$(iii) \Rightarrow (iv)$. By Lemma 2.48 there is an epimorphism $\rho : L \longrightarrow P$ with L a free DG module and an $R^\natural-$linear map $\sigma : p^\natural \longrightarrow l^\natural$ with $\rho\sigma = 1_P$ by hypothesis.
$(iv) \Rightarrow (v) \Rightarrow (i)$ hold by 2.53. □

Homotopically projective DG module The properties of homotopically projective DG modules are summarized in the next theorem.

Theorem 2.59. *[4, 9.5.1] For a DG* $R-$*module* P *the following are equivalent.*

i. P *is homotopically projective.*

i′. $Hom_R(P,-)$ *transforms surjective quasi-isomorphisms into quasi-isomorphisms.*

i″. $Hom_R(P,-)$ *preserves quasi-trivial modules.*

ii. *If* $\alpha : P \longrightarrow N$ *is a chain map and* $\beta : M \longrightarrow N$ *is a quasi-isomorphism, then there is a chain map* $\gamma : P \longrightarrow M$ *such that* $\alpha \sim \beta\gamma$. *Moreover, if* $\gamma' : P \longrightarrow M$ *is a chain map with* $\alpha \sim \beta\gamma'$ *then* $\gamma' \sim \gamma$.

ii′. *If* $\alpha : P \longrightarrow N$ *is a morphism and* $\beta : M \longrightarrow N$ *is a quasi-isomorphism, then there is a morphism* $\gamma : P \longrightarrow M$ *such that* $\alpha \sim \beta\gamma$.

iii. *If* $\rho : M \longrightarrow P$ *is a quasi-isomorphism, then there is a morphism* $\sigma : P \longrightarrow M$ *such that* $\rho\sigma \sim 1_P$*.*

iv. *There exists a semi-projective DG module* Q *and morphisms* $P \xrightarrow{\sigma} Q \xrightarrow{\rho} P$ *such that* $\rho\sigma \sim 1_P$*.*

v. P *is homotopy equivalent to a semi-projective DG module* Q*.*

Proof. $(i) \Rightarrow (i')$ is obvious.
$(i') \Rightarrow (i'')$. If $H(E) = 0$ then the surjective quasi-isomorphism $E \longrightarrow 0$ yields a quasi-isomorphism $Hom_R(P, E) \longrightarrow 0$, and therefore $H(Hom_R(P, E)) = 0$.
$(i'') \Rightarrow (i)$ has been proved in 2.55.
$(i) \Rightarrow (ii)$. Without loss of the generality we can assume that $|\alpha| = 0$. The surjectivity of

$$H_0(Hom_R(P, \beta)) : H_0(Hom_R(P, M)) \longrightarrow H_0(Hom_R(P, N))$$

shows that there exists a morphism $\gamma : P \longrightarrow M$ with $H_0(\beta)[\gamma] = [\alpha]$, so $\beta\gamma = \alpha + \partial(\xi)$ for some $\xi \in Hom_R(P, N)_1$ which means $\beta\gamma \sim \alpha$. If $\alpha \sim \beta\gamma'$, the injectivity of $H_0(\beta)$ shows $[\gamma'] = [\gamma]$. Hence $xg' = \gamma + \partial(\xi)$ for some $\xi \in Hom_R(P, M)_1$, that is, $\gamma' \sim \gamma$.
$(ii) \Rightarrow (iii)$. Apply the hypothesis to $\beta = \rho$ and $\alpha = 1_P$.
$(iii) \Rightarrow (v)$. Let $\rho : Q \longrightarrow P$ be a semi-projective resolution and let $\sigma : P \longrightarrow Q$ be a morphism with $\rho\sigma \sim 1_P$. As $H(\rho)H(\sigma) = 1_{H(P)}$, σ is a quasi-isomorphism. By Lemma 2.57 there is a morphism $\rho' : Q \longrightarrow P$ with $\sigma\rho' \sim 1_Q$ and therefore σ is a homotopy equivalence.
$(v) \Rightarrow (iv)$ is obvious.
$(iv) \Rightarrow (ii')$. Since Q is semi-projective we can apply Lemma 2.57 to the quasi-isomorphism $\beta : M \longrightarrow N$ and the morphism $\alpha\rho : Q \longrightarrow N$. Hence there is a morphism $\gamma' : Q \longrightarrow M$ such that the square in the diagram

$$\begin{tikzcd}
 & Q \arrow[r, "\gamma'"] \arrow[d, "\rho"] & M \arrow[d, "\beta"] \\
P \arrow[ur, "\sigma"] \arrow[r, "1_P"'] & P \arrow[r, "\alpha"'] & N
\end{tikzcd}$$

commutes up to homotopy. Thus, $\gamma = \gamma'\sigma$ satisfies $\beta\gamma = \beta\gamma'\sigma \sim \alpha\rho\sigma \sim \alpha$.

$(ii') \Rightarrow (i'')$. Let E be a DG module with $H(E) = 0$. An element $\alpha \in Z_i(Hom_R(P, E))$ is a morphism $\alpha : P \longrightarrow \Sigma^{-i}E$. For the quasi-isomorphism $\beta : 0 \longrightarrow \Sigma^{-i}E$ there is a morphism $\gamma : P \longrightarrow 0$ with $\alpha \sim \beta\gamma$ by hypothesis, and therefore $\alpha \sim 0$. This means $\alpha \in B_i(Hom_R(P, E))$, so we get $H_i(Hom_R(P, E)) = 0$.

□

Semi-projective DG module

Theorem 2.60. *[4, 9.6.1] For a DG $R-$module P the following are equivalent.*

- $i.$ *P is semi-projective.*
- $i'.$ *$Mor_R(P, -)$ transforms surjective quasi-isomorphisms into surjections.*
- $ii.$ *If $\alpha : P \longrightarrow N$ is a chain map and $\beta : M \longrightarrow N$ is a surjective quasi-isomorphism, then there is a chain map $\gamma : P \longrightarrow M$ such that $\alpha = \beta\gamma$. Moreover, if $\gamma' : P \longrightarrow M$ is a chain map with $\alpha \sim \beta\gamma'$ then $\gamma' \sim \gamma$.*
- $ii'.$ *If $\alpha : P \longrightarrow N$ is a morphism and $\beta : M \longrightarrow N$ is a surjective quasi-isomorphism, then there is a morphism $\gamma : P \longrightarrow M$ such that $\alpha = \beta\gamma$.*
- $iii.$ *Each surjective quasi-isomorphism $\rho : M \longrightarrow P$ has a right inverse.*
- $iv.$ *P is a direct summand of some semi-free DG $R-$module L.*
- $v.$ *P is homotopically projective and linearly projective.*
- $v'.$ *P is homotopically projective and the graded $R^\natural-$module $P^\natural$ is projective.*

Proof. $(i) \Rightarrow (ii')$ has been proved in 2.57.

$(ii') \Rightarrow (iii)$. Any lifting σ of 1_P over ρ is a right inverse of ρ.

$(iii) \Rightarrow (iv)$. Suppose $\rho : L \longrightarrow P$ is the semi-free resolution of P. By assumption ρ has a right inverse, so P is isomorphic to direct summand of L.

$(iv) \Rightarrow (v')$. A semi-free DG module L is homotopically projective by 2.56 and this property passes to its summand by 2.51. Moreover, the graded $R^\natural-$module $L^\natural$ is free, so its direct summand $P^\natural$ is projective.

$(v') \Rightarrow (v)$ is clear.

$(v) \Rightarrow (i)$ follows from the definition.

$(ii', v) \Rightarrow (ii)$. Without loss of the generality we can assume that $|\alpha| = 0$ because $Z_i(Hom_R(P, \beta)) \cong Mor_R(P, \Sigma^{-i}\beta)$. Hence there is a chain map $\gamma : P \longrightarrow M$ with $\alpha = \beta\gamma$. Since P is homotopically projective the by (v),γ is unique up to homotopy by 2.59.

$(ii) \Rightarrow (ii')$ is obvious.

$(ii') \Rightarrow (i')$ by definition. □

Categorically projective DG module We describe the projectivity in $\mathcal{DGM}(R)$.

Definition 2.61. A DG module $P \in \mathcal{DGM}(R)$ is projective (categorically projective) if the functor

$$Mor_R(P, -) : \mathcal{DGM}(R) \longrightarrow \mathcal{M}(K)$$

transforms surjective morphisms into surjections.

Theorem 2.62. *For a DG $R-$module P the following are equivalent.*

i. P is projective.

i′. $Hom_R(P, -)$ transforms surjective morphisms into surjective quasi-isomorphisms.

ii. If $\alpha : P \longrightarrow N$ is a chain map and $\beta : M \longrightarrow N$ is a surjective morphism, then there is a chain map $\gamma : P \longrightarrow M$ such that $\alpha = \beta\gamma$.

ii′. If $\alpha : P \longrightarrow N$ is a morphism and $\beta : M \longrightarrow N$ is a surjective morphism, then there is a morphism $\gamma : P \longrightarrow M$ such that $\alpha = \beta\gamma$.

iii. Each surjective morphism $\rho : M \longrightarrow P$ has a right inverse.

iv. P is a direct summand of some free DG $R-$module L.

v. P is linearly projective and contractible.

v′. P is semi-projective and quasi-trivial.

Proof. $(i') \Rightarrow (i)$. If $\beta : M \longrightarrow N$ is a surjective morphism in $\mathcal{DGM}(R)$ then by hypothesis the induced map

$$Hom_R(P, \beta) : Hom_R(P, M) \longrightarrow Hom_R(P, N)$$

is a surjective quasi-isomorphism and therefore by 2.9 the map

$$Z_0(Hom_R(P,\beta)) : Z_0(Hom_R(P,M)) \longrightarrow Z_0(Hom_R(P,N))$$

is a surjection.

$(i) \Rightarrow (ii')$ by definition of the functor $Mor_R(P,-)$.

$(ii') \Leftrightarrow (ii)$ as $Z_i(Hom_R(P,-)) = Mor_R(P,\Sigma^{-i},-)$.

$(ii') \Rightarrow (iii)$. Any lifting $\gamma : P \longrightarrow M$ of 1_P over ρ satisfies $\rho\gamma = 1_P$.

$(iii) \Rightarrow (iv)$. Assume $\pi : L \longrightarrow P$ is the free resolution of P. By hypothesis there exists a morphism $\sigma : P \longrightarrow L$ with $\pi\sigma = 1_P$. Thus, P is isomorphic to a direct summand of L.

$(iv) \Rightarrow (v')$. By 2.51 it suffices to show that L is semi-projective and quasi-trivial which is immediate.

$(v') \Rightarrow (v)$. As P is semi-projective Theorem 2.60 shows that it is linearly projective and homotopically projective. Thus, $Hom_R(P,-)$ preserves quasi-triviality by 2.55, especially by assumption we have

$$H_0(Hom_R(P,P)) = 0$$

which means that $1_p \in Mor_R(P,P)$=$Z_0(Hom_R(P,P))$ is homotopic to zero by 2.14 and therefore P is contractible.

$(v) \Rightarrow (i)$. Theorem 2.58 and contractibility of P show the result. □

2.7 Injectivity

In the previous section, the concept of projectivity in $\mathcal{DGM}(R)$ was analyzed and it was seen how different the situation is in comparison to ordinary modules. Now, it is quite reasonable if one expects that a dual argument also could be conducted. In the current section, injectivity conditions in $\mathcal{DGM}(R)$ are described. Note that to prove the main theorems of this section a dual argument of their projective counterpart is almost valid and therefore some arguments are omitted.

Definition 2.63. A DG $R-$module I is said to be

- *linearly injective* if $Hom_R(-, I)$ sends injective morphisms into surjections.
- *homotopically injective* if $Hom_R(-, I)$ preserves quasi-isomorphisms.
- *semi-injective* if $Hom_R(-, I)$ transforms injective quasi-isomorphisms into surjective quasi-isomorphisms.

Remark 2.64. For I in $\mathcal{DGM}(R)$ the following hold.

i. I is respectively, linearly injective, homotopically injective and semi-injective over R if $\Sigma^i P$ has the corresponding property for some $i \in \mathbb{Z}$, if and only if $\Sigma^i P$ has that property for all $i \in \mathbb{Z}$.

ii. If $I = \prod_{u \in U} I^u$ then I is respectively, linearly injective, homotopically injective and semi-injective over R if and only if I^u has the corresponding property for every $u \in U$.

Recall that a $K-$module W is faithfully injective if it is injective and $Hom_K(N, W) \neq 0$ for all $K-$modules $N \neq 0$.

Definition 2.65. For a fixed faithfully injective module I (e.g. $Hom_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$) and for an arbitrary complex M of $K-$modules the complex

$$M^{\vee_I} = Hom_K(M, I)$$

of $K-$modules is called the complex of characters of M. If no confusion will arise we denote it just by $M^\vee$. Clearly the assignment $M \mapsto M^\vee$ defines a functor

$$\vee : \mathcal{DGM}(K)^{op} \longrightarrow \mathcal{DGM}(K)$$

which is called character functor.

Lemma 2.66. *There exist natural isomorphisms*

$$Hom_{R^o}(L, M^\vee) \cong Hom_R(M, L^\vee) \cong (L \otimes_R M)^\vee$$

of functors $\mathcal{DGM}(R^o)^{op} \times \mathcal{DGM}(R)^{op} \longrightarrow \mathcal{GM}(K)$ *, and a natural isomorphism*

$$Mor_{R^o}(L, M^\vee) \cong Mor_R(M, L^\vee)$$

of functors $\mathcal{DGM}(R^o)^{op} \times \mathcal{DGM}(R)^{op} \longrightarrow \mathcal{M}(K)$ *.*

Proof. The first isomorphism is given by 2.23 and the second one by adjointness 2.24. Additionally, for the $K-$module of $0-$cycles we have the isomorphism

$$Z_0(Hom_{R^o}(L, M^\vee)) \cong Z_0(Hom_R(M, L^\vee))$$

which shows the last isomorphism.

□

Character functor is a very useful tool in analyzing injectivity. The next proposition is our start point for discovering properties of injective DG modules.

Proposition 2.67. *Let P be a graded A^o-module or a DG R^o-module. If P is projective over A^o (respectively linearly projective, homotopically projective, semi-projective over R^o), then its character module $P^\vee$ is injective over A (respectively linearly injective, homotopically injective, semi-injective over R^o).*

Proof. By Lemma 2.66 there exists an isomorphism of functors

$$Hom_R(-, P^\vee) \cong Hom_{R^o}(P, (-)^\vee)$$

It is not hard to see that if β is injective then $\beta^\vee$ is surjective and if it is quasi-isomorphism then $\beta^\vee$ is a quasi-isomorphism as well. Now a simple comparison between various definitions of projectivity and injectivity yields the result. □

Definition 2.68. Let I be a graded $A-$module. I is injective if the functor

$$Hom_A(-, I) : \mathcal{GM}(A) \longrightarrow \mathcal{GM}(K)$$

transforms injective morphisms into surjections.

Proposition 2.69. *For a graded $A-$module I the following are equivalent.*

i. *I is injective.*

ii. *$Hom_A(-, I)$ preserves exact sequences.*

iii. *If $\alpha : M \longrightarrow I$ is a homomorphism and $\beta : M \longrightarrow N$ is a injective morphisms then there is a homomorphism $\gamma : N \longrightarrow I$ such that $\gamma\beta = \alpha$ which means the diagram below commutes.*

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & I \\ N \arrow[ur, dotted, "\gamma"'] & \end{tikzcd}$$

iv. If $\sigma : I \longrightarrow N$ *is an injective morphism, then there is a morphism* $\rho : N \longrightarrow I$ *such that* $\rho\sigma = 1_I$.

v. I *is a direct summand of* $L^\vee$ *for some free graded* A^o-*module* L.

Proof. $(i') \Leftrightarrow (i'')$ holds because $Hom_R(-, I)$ is left exact.
$(i) \Leftrightarrow (ii')$ and $(i') \Leftrightarrow (ii)$ is almost clear.
$(ii') \Leftrightarrow (ii)$ as $Hom_A(M, I)_i = Mor_A(\Sigma^i M, I)$ for $i \in \mathbb{Z}$.
$(ii') \Rightarrow (iii)$. By assumption, for $\alpha = 1_I$ and $\beta = \sigma$ there is a morphism $\rho : N \longrightarrow I$ such that $\rho\sigma = 1_I$.
$(iii) \Rightarrow (iv)$. By dualizing a free resolution of $I^\vee$ it is easy to see that there is an injective morphism $\sigma : I \longrightarrow L^\vee$, where L is a free graded module over A^o. By hypothesis there is a morphism $\rho : L^\vee \longrightarrow I$ such that $\rho\sigma = 1_I$, so I is isomorphic to a direct summand of $L^\vee$.
$(iv) \Rightarrow (i)$. Remark 2.64 and 2.67 yield the result. □

Proposition 2.70. *Let* $(\pi^{u,u-1} : I^u \longrightarrow I^{u-1})_{u\in\mathbb{Z}}$ *be an inverse system of epimorphisms of graded* $A-$*modules with* $I^u = 0$ *for* $u \ll 0$. *If* $K^u = Ker(\pi^{u,u-1})$ *is injective for all* $u \in \mathbb{Z}$, *then* $I = \varprojlim_u I^u$ *is injective and* $I \cong \prod_{u\in\mathbb{Z}} K^u$.

Proof. Set $J = \prod_{u\in\mathbb{Z}} K^u$, and consider I as a submodule of $\prod_{u\in\mathbb{Z}} I^u$. By Proposition 2.69 we can choose for each u an $A-$linear map $\sigma^{u-1} : I^{u-1} \longrightarrow I^u$ with $\pi^{u,u-1}\sigma^{u-1} = 1_{I^{u-1}}$. Then for $(\cdots, x^u, \cdots) \in I$ we have the equality

$$\pi^{u,u-1}(x^u - \sigma^{u-1}(x^{u-1})) = \pi^{u,u-1}(x^u) - x^{u-1} = 0$$

Thus the assignment

$$(\cdots, x^u, \cdots) \mapsto (\cdots, x^u - \sigma^{u-1}(x^{u-1}), \cdots)$$

defines a map $I \mapsto J$.
If $y = (\cdots, y^u, \cdots)$ is a non-zero element of J, then the number $v = inf\{u | y^u \neq 0\}$ is finite, and therefore

$$\tau^u(y) = y^u + (\sigma^{u-1}(y^{u-1})) + \cdots + (\sigma^{u-1} \cdots \sigma^{v+1}\sigma^v(y^v))$$

is a well defined element of I^u. We set $\tau^u(0) = 0$ and note that $\pi^u(y^u) = 0$ implies $\pi^u\tau^u(y) = \tau^{u-1(y)}$ for all $u \in \mathbb{Z}$. Hence, the assignment

$$(\cdots, y^u, \cdots) \mapsto (\cdots, \tau^u(y^u), \cdots)$$

defines a maps $J \mapsto I$. It is easy to see that the two maps defined above are inverse isomorphisms and therefore I is injective. $\square$

Although the argument for proving the next lemma is exactly the dual of the argument for 2.55, a proof to show how to deal with the other statements of this section is provided here.

Lemma 2.71. *Let J be a DG module over R.*

i. J is homotopically injective if and only if $Hom_R(-, J)$ preserves quasi-triviality.

ii. If $J = \varprojlim_u J^u$ for an inverse system $(\ \pi^{u,u-1} : J^u \longrightarrow J^{u-1}\ )_{u\in\mathbb{Z}}$ of epimorphisms of DG $R-$modules with $J^u = 0$ for $u \ll 0$, $K^u = Ker(\pi^{u,u-1})$ is homotopically injective for all $u \in \mathbb{Z}$, and $(K^u)^\natural$ is injective over $R^\natural$ for all u, then J is homotopically injective and $J^\natural$ is injective over $R^\natural$.

Proof. (i) Assume first J is homotopy injective. If E is a quasi-trivial DG module, then $0 \longrightarrow E$ is a quasi-isomorphism, hence so is $Hom_R(E, J) \longrightarrow 0$.
Assume next $Hom_R(-, J)$ preserves quasi-triviality. If $\beta : M \longrightarrow N$ is a quasi-isomorphism, then the mapping cone $C = Cone\ \beta$ is quasi-trivial. Therefore $Hom_R(C, J)$, isomorphic to $\Sigma^{-1}Cone\ \beta'$ where $\beta' = Hom_R(\beta, J)$, is quasi-trivial which means β' is a quasi-isomorphism.
(ii) Proposition 2.70 shows that $J^\natural$ is injective over $R^\natural$, so by (i) it suffices to prove that if $H(E) = 0$, then $H(Hom_R(E, J)) = 0$. For each $u \in \mathbb{Z}$ the sequence

$$0 \longrightarrow K^u \longrightarrow J^u \longrightarrow J^{u-1} \longrightarrow 0$$

of DG modules is linearly split, so the induced sequence of complexes

$$0 \longrightarrow Hom_R(E, K^u) \longrightarrow Hom_R(E, J^u) \longrightarrow Hom_R(E, J^{u-1}) \longrightarrow 0$$

is exact. As $J^u = 0$ for $u \ll 0$, by induction on u, we may assume $H(Hom_R(E, J^{u-1}))$ is quasi-trivial. By hypothesis K^u is homotopically injective, so $H(Hom_R(E, K^u)) = 0$ by (i). Thus, $H(Hom_R(E, J^u)) = 0$ and therefore

$$H(Hom_R(E, J)) = H(Hom_R(E, \varprojlim_u J^u)) \cong H(\varprojlim Hom_R(E, J^u)) = 0$$

which leads to the desired result. □

Lemma 2.72. *Let J be a semi-injective DG module, $\alpha : M \longrightarrow J$ be a morphism of DG modules, and $\beta : M \longrightarrow N$ be a quasi-isomorphism of DG modules.*

i. If β is injective, then there exists a morphism $\gamma : N \longrightarrow J$ with $\gamma\beta = \alpha$.

ii. There exists a morphism $\gamma : N \longrightarrow J$ such that $\gamma\beta \sim \alpha$.

Proof. By hypothesis $Hom_R(\beta, J)$ is a surjective quasi-isomorphism, so

$$Mor_R(\beta, J) = Z_0 Hom_R(\beta, J) : Mor_R(N, J) \longrightarrow Mor_R(M, J)$$

is surjective by 2.9, hence there is a morphism $\gamma : N \longrightarrow J$ with $\gamma\beta = \alpha$. For (ii) a dual of the argument for part (ii) of 2.57 leads to the result, considering the fact that we need to apply the last part of 2.12. □

Linearly injective DG module

Theorem 2.73. *[4, 10.4.1] For a DG $R-$module I the following are equivalent.*

i. I is linearly injective.

i′. $Mor_{R^\natural}(-, I^\natural)$ preserves exact sequences.

i″. $Hom_R(-, I)$ preserves exact sequences.

ii. If $\alpha : M \longrightarrow I$ is a chain map and $\beta : M \longrightarrow N$ is an injective morphism, then there is a homomorphism $\gamma : N^\natural \longrightarrow I^\natural$ in $\mathcal{GM}(R^\natural)$ with $\gamma\beta = \alpha$.

ii′. If $\alpha : M \longrightarrow I$ is a morphism and $\beta : M \longrightarrow N$ is an injective morphism, then there is a morphism $\gamma : N^\natural \longrightarrow I^\natural$ in $\mathcal{GM}(R^\natural)$ with $\gamma\beta = \alpha$.

iii. If $\sigma : I \longrightarrow N$ is an injective morphism then there exists a morphism $\rho : N^\natural \longrightarrow I^\natural$ in $\mathcal{GM}(R^\natural)$ such that $\rho\sigma = 1_I$.

iv. *I is a DG submodule of $L^\vee$ for some free DG module L over R^o, with $I^\natural$ a direct summand of $(L^\vee)^\natural$ in $\mathcal{GM}(R^\natural)$.*

v. *$I^\natural$ is an injective graded $R^\natural-$module.*

Proof. A dual of the argument for 2.58 works for all parts except $(iii) \Leftrightarrow (iv)$. Let $\pi : L \longrightarrow I^\vee$ be the free resolution of $I^\vee$ over R^o, also let $\sigma : I \longrightarrow L^\vee$ be the composition of $\pi^\vee$ and the natural map $f : I \longrightarrow I^{\vee\vee}$. As σ is injective then by hypothesis there is a morphism $\rho : (L^\natural)^\vee \longrightarrow I^\natural$ of graded $R^\natural-$modules with $\sigma\rho = 1_I$.

$(iv) \Leftrightarrow (v) \Leftrightarrow (i)$ hold by 2.69.

□

Homotopically injective DG module A dual of the argument for 2.59 proves the following theorem.

Theorem 2.74. *[4, 10.5.1] For a DG $R-$module I the following are equivalent.*

i. *I is homotopically injective.*

i′. *$Hom_R(-, I)$ transforms injective quasi-isomorphisms into quasi-isomorphisms.*

i″. *$Hom_R(-, I)$ preserves quasi-trivial modules.*

ii. *If $\alpha : M \longrightarrow I$ is a chain map and $\beta : M \longrightarrow N$ is a quasi-isomorphism, then there is a chain map $\gamma : N \longrightarrow I$ such that $\gamma\beta \sim \alpha$. Moreover, if $\gamma' : N \longrightarrow I$ is a chain map with $\gamma'\beta \sim \alpha$ then $\gamma' \sim \gamma$.*

ii′. *If $\alpha : M \longrightarrow I$ is a morphism and $\beta : M \longrightarrow N$ is a quasi-isomorphism, then there is a morphism $\gamma : N \longrightarrow I$ such that $\gamma\beta \sim \alpha$.*

iii. *If $\sigma : I \longrightarrow N$ is a quasi-isomorphism, then there is a morphism $\rho : N \longrightarrow I$ such that $\rho\sigma \sim 1_I$.*

iv. *There exists a semi-injective DG module J and morphisms $I \xrightarrow{\sigma} J \xrightarrow{\rho} I$ such that $\rho\sigma \sim 1_I$.*

v. *I is homotopy equivalent to a semi-injective DG module J.*

Semi-injective DG module

Theorem 2.75. *[4, 10.6.1] For a DG $R-$module I the following are equivalent.*

i. *I is semi-injective.*

i′. *$Mor_R(-, I)$ transforms injective quasi-isomorphisms into surjections.*

ii. *If $\alpha : M \longrightarrow I$ is a chain map and $\beta : M \longrightarrow N$ is an injective quasi-isomorphism, then there is a chain map $\gamma : N \longrightarrow I$ such that $\gamma\beta = \alpha$. Moreover, if $\gamma' : N \longrightarrow I$ is a chain map with $\gamma'\beta \sim \alpha$ then $\gamma' \sim \gamma$.*

ii′. *If $\alpha : M \longrightarrow I$ is a morphism and $\beta : M \longrightarrow N$ is an injective quasi-isomorphism, then there is a morphism $\gamma : N \longrightarrow I$ such that $\gamma\beta = \alpha$.*

iii. *Each injective quasi-isomorphism $\sigma : I \longrightarrow M$ has a left inverse.*

iv. *I is homotopically injective and linearly injective.*

iv′. *I is homotopically injective and the graded $R^\natural-$module $I^\natural$ is injective.*

It is true that a dual argument of the 2.60 can be conducted almost for all parts of this theorem but we need the concept of semi-injective resolution which will be given after defining an injective model on $\mathcal{DGM}(R)$. However it is possible to talk about semi-injective resolution directly, in fact, it is possible to construct a semi-injective resolution for a DG module; c.f 3.37.

Categorically injective DG module We describe the injectivity in $\mathcal{DGM}(R)$.

Definition 2.76. A DG module $I \in \mathcal{DGM}(R)$ is injective (categorically injective) if the functor

$$Mor_R(-, I) : \mathcal{DGM}(R)^{op} \longrightarrow \mathcal{M}(K)$$

transforms injective morphisms into surjections.

The next theorem is dual of theorem 2.62 and its proof is dual of the argument for 2.62 which is omitted here.

Theorem 2.77. *For a DG $R-$module I the following are equivalent.*

i. *I is injective.*

i'. $Hom_R(-,I)$ transforms injective morphisms into surjective quasi-isomorphisms.

ii. If $\alpha : M \longrightarrow I$ is a chain map and $\beta : M \longrightarrow N$ is an injective morphism, then there is a chain map $\gamma : N \longrightarrow I$ such that $\gamma\beta = \alpha$.

ii'. If $\alpha : M \longrightarrow I$ is a morphism and $\beta : M \longrightarrow N$ is an injective morphism, then there is a morphism $\gamma : N \longrightarrow I$ such that $\gamma\beta = \alpha$.

iii. Each injective morphism $\sigma : I \longrightarrow N$ has a left inverse.

iv. I is a direct summand of $L^\vee$ for some free DG module L over R^o.

v. I is linearly injective and contractible.

v'. I is semi-injective and quasi-trivial.

2.8 Flatness

Definition 2.78. A DG $R-$module F is said to be

- *linearly flat* if $(F \otimes_R -)$ preserves injective morphisms.
- *homotopically flat* if $(F \otimes_R -)$ preserves quasi-isomorphisms.
- *semi-flat* if $(F \otimes_R -)$ preserves injective quasi-isomorphisms.
- *flat* if $(F \otimes_R -)$ transforms injective morphisms into injective quasi-isomorphism.

Remark 2.79. For F in $\mathcal{DGM}(R)$ the following hold.

i. F is respectively, linearly flat, homotopically flat and semi-flat over R if $\Sigma^i F$ has the corresponding property for some $i \in \mathbb{Z}$, and if only if $\Sigma^i F$ has that property for all $i \in \mathbb{Z}$.

ii. If $F = \coprod_{u \in U} F^u$, then F is respectively, linearly flat, homotopically flat and semi-flat over R if and only if F^u has the corresponding property for every $u \in U$.

Definition 2.80. Let F be a graded $A-$module. F is flat if the functor

$$(F \otimes_A -) : \mathcal{GM}(A) \longrightarrow \mathcal{GM}(K)$$

preserves injective morphisms.

The next proposition says that, under colimits, flatness behaves better than projectivity.

Proposition 2.81. *If $\mathbb{F}$ is a filtered direct system of flat graded A^o-modules, then the graded A^o-module $F = \varinjlim \mathbb{F}$ is flat.*
If $\mathbb{F}$ is a filtered direct system of linearly flat (respectively, homotopically flat, semi-flat, flat) DG modules over R^o, then the DG module $F = \varinjlim \mathbb{F}$ has the corresponding property.

Proof. The isomorphism

$$\varinjlim(\mathbb{F} \otimes_R M) \cong (\varinjlim \mathbb{F}) \otimes_R M$$

along with 2.35 and 2.36 show the result. □

A convenient way to find properties of flatness for DG modules is to deduce them from available properties of injectivity, using the next proposition.

Proposition 2.82. *A graded A^o-module F is flat if and only if for some (equivalently, every) character functor $\vee$ the graded $A-$module $F^\vee$ is injective.*
A DG module F over R^o is linearly flat (respectively, homotopically flat, semi-flat, flat) if and only if for some (equivalently, every) character functor $\vee$ the DG module $F^\vee$ over R is linearly injective (respectively, homotopically injective, semi-injective, injective).

Proof. Recall that a morphism β in $\mathcal{GM}(A)$ or in $\mathcal{DGM}(R)$ is injective if and only if $\beta^\vee$ is surjective, and is a quasi-isomorphism if and only if $\beta^\vee$ is a quasi-isomorphism. The adjointness isomorphisms 2.24 yields isomorphism of functors

$$(F \otimes_A -) \cong Hom_A(-, F^\vee) : \mathcal{GM}(A)^{op} \longrightarrow \mathcal{GM}(K)$$

$$(F \otimes_R -) \cong Hom_R(-, F^\vee) : \mathcal{DGM}(R)^{op} \longrightarrow \mathcal{DGM}(K)$$

Now a series of comparisons between flatness and injectivity lead to the desired results. □

Combining proposition 2.67 and 2.82 gives us the next proposition.

Proposition 2.83. *Every linearly projective (respectively, homotopically projective, semi-projective, projective) DG module over R^o is linearly flat (respectively, homotopically flat, semi-flat, flat).*

Theorem 2.84. *[4, 11.2.1] Let F be a DG module over R^o.*

i. F is linearly flat if and only if the graded $R^{o\natural}-$module $F^\natural$ is flat.

ii. F is homotopically flat if and only if $(F \otimes_R -)$ preserves quasi-triviality.

iii. F is homotopically flat if it is homotopically equivalent to a semi-flat DG module.

iv. F is semi-flat if and only if it is linearly flat and homotopically flat.

v. F is flat if and only if it is semi-flat and quasi-trivial.

Proof. (i) and (iv) are consequence of Proposition 2.82 and the corresponding part of Theorems 2.73 and 2.75.
(ii) and (v) result from Proposition 2.82 and the corresponding part of Theorems 2.74 and 2.77.
(iii). If F is homotopy equivalent to a semi-flat DG module F', then complexes $F \otimes_R E$ and $F' \otimes_R E$ are homotopy equivalent for every DG $R-$module E, in particular $H(F \otimes_R E) \cong H(F' \otimes_R E)$. By (iv), F' is homotopically flat so if $H(E) = 0$, then $H(F' \otimes_R E) = 0$. Hence $H(F \otimes_R E) = 0$ which means F is homotopically flat by (ii). □

2.9 Finiteness

In this section, a language which describes finiteness properties for DG objects will be developed. Note that if M is in $\mathcal{DGM}(R)$, then finiteness hypotheses may appear either as properties of the underlying graded module $M^\natural$ over the graded algebra $R^\natural$, or as properties of the graded homology module $H(M)$ over the graded algebra $H(R)$.

Finiteness for graded modules Finiteness conditions on modules are usually imposed in terms of number of generators. There are two ways to implement such an approach for DG objects. One is to do it separately in each degree, regarding M_n as a module over R_0. The other one is to impose the conditions globally on M.

Definition 2.85. Let M be a graded $A-$module.

- M is degreewise finite if it admits a set of generators X, with X_i finite for each $i \in \mathbb{Z}$, where X_i is the set of elements of degree i in X.
- M is finite if it admits a set of generators X, with X finite.
- M is noetherian if each graded submodule of M is finite.
- The graded algebra A is left (respectively, right) noetherian if the graded $A-$module (respectively A^o-module) A has the corresponding property.
- M is finitely presented if the functor $Hom_A(M, -)$ preserves direct limits.

Remark 2.86. Comparing the definitions of finitely presented and $\kappa-$small object shows that finitely presented modules are $\aleph_0-$small.

In the above definition, the distribution of non-zero components in a graded object has important consequences for its finiteness.

Definition 2.87. Let M be a graded module, We set

$$inf\ M = inf\{i \in \mathbb{Z} | M_i \neq 0\} \tag{2.9.1}$$

$$sup\ M = sup\{i \in \mathbb{Z} | M_i \neq 0\} \tag{2.9.2}$$

- M is *bounded below* if $inf\ M > -\infty$.
- M is *bounded above* if $sup\ M < \infty$.
- M is *bounded* if it is bounded both below and above.
- The graded algebra A with $A_i = 0$ for all $i < 0$ is said to be non-negatively graded.

Proposition 2.88. *Let M be a graded module over a graded algebra A.*

i. If M is finite and A is left noetherian, then M is noetherian.

ii. If M is degreewise finite over A_0 and A_0 is left noetherian, then M_i is noetherian over A_0 for each $i \in \mathbb{Z}$.

Proposition 2.89. *Let M be a graded module over a graded algebra A and A is non-negatively graded, the following hold.*

i. If M is finite, then it is bounded below.

ii. If M is non-trivial, bounded below, and degreewise finite over A, then for $j \in \mathbb{N}$ the A_0-module M_j is finite.

iii. If M is bounded below and degreewise finite over A, and A is degreewise finite over A_0, then M is degreewise finite over A_0.

iv. If M is noetherian, then M_i is noetherian over A_0 for each $i \in \mathbb{Z}$.

Proof. Let X be a set of generators of M. First of all, note that there is a surjective morphism $\coprod_{x \in X} Ae_x \longrightarrow M$ with $|e_x| = |x|$ for all $x \in X$.
(i) If X is finite for $h < \inf\{|x| : x \in X\}$ we have $M_h = 0$.
(ii) If $j = inf\ M$ then $M_j = A_0 X_j$ and therefore the A_0-module M_j is finite because X_j is finite.
(iii) Choose X with X_i finite for each $i \in \mathbb{Z}$ and empty for $i < j$ where $j = inf\ M$. For each $h \in \mathbb{Z}$ the A_0-module M_h is a homomorphic image of the A_0-module $\coprod_{j \leq |e_x| \leq h} A_{h-|e_x|}$ which is finite by hypothesis.
(iv) As A is non-negatively graded, the $K-$submodule $M_{\geq i}$ is a graded $A-$submodule of M for each $i \in \mathbb{Z}$. Because A is noetherian, $M_{\geq i}$ is finite over A, and therefore so is $M_{\geq i}/M_{\geq i+1}$. Now, $A_{\geq 1}$ annihilates $M_{\geq i}/M_{\geq i+1}$, so $M_{\geq i}/M_{\geq i+1}$, which is isomorphic to M_i, is finite over $A/A_{\geq 1} \cong A_0$. $\square$

Homological finiteness

Definition 2.90. Let R be a DG algebra and M be a DG module. If $R^\natural$ (respectively, $M^\natural$) has one of the finiteness properties of Definition 2.85 or 2.87, then we say R (respectively, M) has the corresponding properties. Furthermore, we say M is homotopically finitely presented if it has a semi-projective resolution $\hat{M}$ such that $\hat{M}$ is finitely presented.

We provide one of the main results regarding finiteness condition in Theorem 2.94. But to prove it some lemmas and definitions are needed which are given first.

Definition 2.91. For a DG module M over a non-negatively DG algebra R, set

$$\tau_{\geq j}(M) = \cdots \longrightarrow M_{j+s} \xrightarrow{\partial_{j+2}} M_{j+1} \longrightarrow Z_j(M) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\tau_{\leq j}(M) = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_j/B_j(M) \longrightarrow M_{j-1} \xrightarrow{\partial_{j-1}} M_{j-2} \longrightarrow \cdots$$

These DG modules are called truncations of M at level j. We also call truncations the canonical morphisms

$$\tau_{\geq j} : \tau_{\geq j}(M) \longrightarrow M \text{ and } \tau_{\leq j} : M \longrightarrow \tau_{\leq j}(M) \tag{2.9.3}$$

which are injective and surjective, respectively.

Remark 2.92. For each DG module M over a non-negatively graded DG algebra R the canonical maps

$$\varinjlim_j(\tau_{\geq j}) : \varinjlim_j(\tau_{\geq j}(M)) \longrightarrow M$$

$$\varprojlim_j(\tau_{\leq j}) : M \longrightarrow \varprojlim_j(\tau_{\leq j}(M))$$

$$\tau_{\geq j} : H_i(\tau_{\geq j}(M)) \longrightarrow H_i(M) \text{ for all } i \geq j$$

$$\tau_{\leq j} : H_i(M) \longrightarrow H_i(\tau_{\leq j}(M)) \text{ for all } i \leq j$$

are bijective. Hence, the truncations give good bounded approximations of the module structure while their homology approximates $H(M)$ at the same time.

A quasi-trivial DG module E is *elementary* if $sup\ E = inf\ E + 1$.

Lemma 2.93. *Each elementary quasi-trivial DG module E over a non-negatively graded DG algebra R is contractible.*

Proof. Suppose $E \;=: \cdots \longrightarrow 0 \longrightarrow E_{i+1} \xrightarrow{\partial_{i+1}} E_i \longrightarrow 0 \longrightarrow \cdots$ for some i. Since E is quasi-trivial ∂_{i+1} is an isomorphism of $K-$modules. In fact it is an isomorphism of R_0-module and R_0-linear as well, because R is non-negatively graded and $\partial(R_0) = 0$. Now, the degree 1 map $\xi : E \longrightarrow E$, defined by $\xi_i = \partial_{i+1}^{-1}$ and $\xi_n = 0$ for $n \neq i$, is a contracting homotopy. $\square$

Theorem 2.94. *[4, 12.3.5] Let R be a non-negatively graded DG algebra. For a DG $R-$module M the following hold.*

i. If $M^\natural$ is free over $R^\natural$ and M is bounded below, then M is semi-free.

ii. If $M^\natural$ is projective over $R^\natural$ and M is bounded below, then M is semi-projective.

iii. If $M^\natural$ is injective over $R^\natural$ and M is bounded above, then M is semi-injective.

iv. If $M^\natural$ is flat over $R^\natural$ and M is bounded below, then M is semi-flat.

Proof. (i) Let M be a bounded below DG module such that the graded $R^\natural-$module $M^\natural$ is free. Choose a basis E of $M^\natural$. For each $e' \in E$, the canonical decomposition $\partial(e') = \sum_{e \in E} r_e e$ with $r_e \in R$ has $r_e = 0$ whenever $|e| \geq |e'|$, due to the fact that R is a non-negatively graded. Therefore, the sets $E^u = \{e \in E \ : \ |e| \leq u + inf \ M\}$ where $u \in \mathbb{Z}$ provide a semi-basis $E = \sqcup_{u \in \mathbb{Z}} E^u$ of M.
(ii) Let M be a bounded below DG module with $M^\natural$ projective over $R^\natural$. By Theorem 2.60 it suffices to prove that M is homotopically projective which is equal to show that for each quasi-trivial DG $R-$module E, $H(Hom_R(M, E)) = 0$ by Theorem 2.59. First, we prove this when E is bounded above. To do this, we fix an integer u and prove $H_u(Hom_R(M, E)) = 0$ by induction on $sup\ E$. This is clear when $sup\ E < u + \inf\ M$, because in that case $Hom_R(M, u)_u = 0$. Hence, we may assume that $H_u(Hom_R(M, E)) = 0$ whenever $sup\ E \leq i$ for some $i \in \mathbb{Z}$. Let E' be a quasi-trivial DG module with $sup\ E' = i + 1$. The exact sequence of DG modules

$$0 \longrightarrow E'' \longrightarrow E' \xrightarrow{\tau_{\leq i}} \tau_{\leq i}(E') \longrightarrow 0$$

defines a DG module E'' which is quasi-trivial with

$$sup\ E'' = i + 1 \text{ and } inf\ E'' = i$$

Since M is linearly projective we get an exact sequence of complexes

$$0 \longrightarrow Hom_R(M, E'') \longrightarrow Hom_R(M, E') \xrightarrow{\tau_{\leq i}} Hom_R(M, \tau_{\leq i}(E')) \longrightarrow 0$$

and hence an exact sequence of homology $K-$modules

$$H_u Hom_R(M, E'') \longrightarrow H_u Hom_R(M, E') \longrightarrow H_u Hom_R(M, \tau_{\leq i}(E'))$$

In the homology exact sequence the first term vanishes because E'' is a contractible elementary module and therefore Lemma 2.93 can be used. The last term vanishes by induction hypothesis. Thus $H_u Hom_R(M, E') = 0$ as desired. Since u was chosen arbitrarily then we have $H_u Hom_R(M, E) = 0$ whenever E is quasi-trivial and bounded above.

Next, we consider any quasi-trivial DG module E. Combining 2.92 and 2.29 yields the isomorphisms of complexes

$$Hom_R(M, E) \cong Hom_R(M, \varprojlim_j \tau_{\leq j}(E)) \cong \varprojlim_j Hom_R(M, \tau_{\leq j}(E))$$

where the first limit is over the surjective morphism $\tau_{\leq j+1}(E) \longrightarrow \tau_{\leq j}(E)$. As M is linearly projective the morphisms $Hom_R(M, \tau_{\leq j+1}(E)) \longrightarrow Hom_R(M, \tau_{\leq j}(E))$, defining the second limit, are surjective. Each $\tau_{\leq j}(E)$ is bounded above so we know that $HHom_R(M, \tau_{\leq j}(E)) = 0$. Because $Hom_R(M, \tau_{\leq j}(E))$ form a quasi-trivial tower of surjective morphisms their limit is quasi-trivial as well.

(iii) A dual argument of (ii) can be applied just note that a descending induction on $inf\ E$ should be considered.

(iv) If M is a bounded below DG module which is linearly flat, then the character module $M^{\vee\natural}$ is injective over $R^{\natural}$. Thus $M^{\vee}$ in injective by (iii) so M is semi-flat. □

We finish this chapter by the next definition which is highly employed in chapter 4.

Definition 2.95. The semi-free filtration $\mathcal{L}$ of the DG module L is finite if $L^u = 0$ for $u \gg 0$ and the basis of L^u/L^{u-1} is a finite set. In this case, L is called a small semi-free DG module. In addition, a semi-projective module P is small semi-projective if it is retract of a small semi-free module.

Remark 2.96. Every small semi-projective module is finite.

3

Model Category and DG-Modules

The main aim of this chapter is to define model structures on the category of DG modules. Through this chapter R is DG algebra and $\mathcal{DGM}(R)$ denotes the category of DG modules over R.

3.1 More on DG-Modules

In this section, we prove more properties of differential graded modules which are used in arguments regarding model structures.

Lemma 3.1. *In $\mathcal{DGM}(R)$, the transfinite composition of injective maps is injective and the transfinite composition of quasi-isomorphisms is a quasi-isomorphisms.*

Proof. **Injections** We claim that the transfinite composition of injections is injective by induction on ordinals. The composition of finitely many injective maps is an injection, so for finite ordinals, the transfinite composition of a $\lambda-$sequence with finite λ is injective.

For an arbitrary ordinal λ and $\lambda-$sequence X_β consider a transfinite composition $\pi_0 : X_0 \longrightarrow \varinjlim_{\beta<\lambda}(X_\beta)$. Now, if $\pi_0(x) = 0$ for some x, then there is a $\alpha < \lambda$ such that $\pi_0^\alpha(x) = 0$. By induction hypothesis π_0^α is an injective map hence $x = 0$ and therefore π_0 is an injection.

Quasi-isomorphism For an ordinal λ, consider a $\lambda-$sequence of quasi-isomorphisms as below in which $\beta < \lambda$ and π_βs are equivalent to μ^β in Definition 2.30.

$$\begin{tikzcd}
X_0 \arrow[r, "i_0"] \arrow[drr, "\pi_0"'] & X_1 \arrow[r, "i_1"] \arrow[dr, "\pi_1"] & X_2 \arrow[r, "i_2"] \arrow[d, "\pi_2"] & \cdots \arrow[r] & X_\beta \arrow[r, "i_\beta"] \arrow[dll, "\pi_\beta"] & \cdots \\
 & & \varinjlim_{\beta<\lambda}(X_\beta) & & &
\end{tikzcd}$$

Applying H_* yields the diagram

$$\begin{tikzcd}
H_*(X_0) \arrow[r, "H_*(i_0)"] \arrow[dr, "\pi_0^\star"] \arrow[ddrr, dotted, "H_*(\pi_0)"'] & H_*(X_1) \arrow[r, "H_*(i_1)"] \arrow[dr, "\pi_1^\star"] \arrow[ddr, dotted, "H_*(\pi_1)"'] & H_*(X_2) \arrow[r, "H_*(i_2)"] \arrow[d, "\pi_2^\star"] & \cdots \arrow[r] & H_*(X_\beta) \arrow[r, "H_*(i_\beta)"] \arrow[dll, "\pi_\beta^\star"'] \arrow[ddll, dotted, "H_*(\pi_\beta)"] & \cdots \\
 & & \varinjlim_{\beta<\lambda} H_*(X_\beta) \arrow[d, "u"] & & & \\
 & & H_*(\varinjlim_{\beta<\lambda}(X_\beta)) & & &
\end{tikzcd} \tag{3.1.1}$$

in which $\pi^\star$s are equivalent to μ^β in Definition 2.30. In diagram 3.1.1 $H_*(i_\beta)$s are isomorphisms and therefore $\pi^\star$s are isomorphisms as well. In addition, 2.36 shows u is bijective. Hence $H_*(\pi_0)$ is an isomorphism and therefore π_0 is a quasi-isomorphism showing that the transfinite composition of quasi-isomorphisms is also a quasi-isomorphism.

□

Lemma 3.2. *In an abelian category, a morphism $g : A \longrightarrow B$ has the right lifting property with respect to all injective maps if and only if g is surjective with injective kernel. In fact, g has a right inverse.*

Proof. **Surjection** As g has the right lifting property with respect to all injective maps, the diagram

$$
\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, hook, "0"'] & A \arrow[d, "g"] \\
B \arrow[r, "1_B"'] \arrow[ur, dotted, "h"] & B
\end{tikzcd}
$$

commutes. As $gh = 1_B$ therefore g has a right inverse and is surjective.

Ker Since g has a right inverse the exact sequence $0 \hookrightarrow Kerg \xrightarrow{i} A \xrightarrow{g} B \longrightarrow 0$ splits. Hence $A \cong B \oplus Kerg$ and i has a left inverse called i'.

The diagram

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, hook, "f"'] & Kerg \\
Y &
\end{tikzcd}
\tag{3.1.2}
$$

in which f is an injective morphism and α is an arbitrary morphism, can be extended to the diagram

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, hook, "f"'] & Kerg \arrow[r, hook, "i"] & A \arrow[d, "g"] \\
Y \arrow[r] \arrow[ur, dotted, "\bar{h}"] \arrow[urr, dotted, "h"] & 0 \arrow[r] & B
\end{tikzcd}
\tag{3.1.3}
$$

In the diagram 3.1.3, h exists due to the lifting property of g and $\bar{h} = i'h$, which is a solution to the diagram 3.1.2.

Lifting Suppose $g : A \longrightarrow B$ is a surjective map with injective kernel. As the kernel is an injective object then $A \cong B \oplus Kerg$. Consider $i : X \longrightarrow Y$ as an arbitrary injective map, and consider the commutative diagram below

$$
\begin{tikzcd}
X \arrow[r, "\alpha=(\alpha_1,\alpha_2)"] \arrow[d, "i"'] & Kerf \oplus B \arrow[d, "\pi_2"] \\
Y \arrow[r, "\beta"'] \arrow[ur, dotted, "h=(h_1,\beta)"] & B
\end{tikzcd}
$$

h_1 exists because $Kerf$ is injective and we have $h_1 i = \alpha_1$. Also $\alpha_2 = \pi_2\alpha = \beta i$ and $hi = h_1 i + \beta i = \alpha$. In addition, $\pi_2 h = \beta$ which shows the diagram commutes and therefore h is a lift. □

Lemma 3.3. *In an abelian category, if a map* $f : X \longrightarrow Y$ *has the right lifting property with respect to the map* $i : A \longrightarrow B$ *then any pullback of* f *has the lifting property as well, specially* $0 : kerf \longrightarrow 0$ *has the lifting property.*

Proof. Consider a pullback diagram for f as below

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, "\alpha'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "\beta'"'] & Y
\end{tikzcd}$$

For a given lifting problem for f' with respect to i, it can be extended to the following diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \times_Y Y' \arrow[r, "\alpha'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
B \arrow[r, "\beta"'] \arrow[ur, bend right, "u"'] \arrow[urr, bend left, "h"] & Y' \arrow[r, "\beta'"'] & Y
\end{tikzcd}$$

in which h is a lift for f and u is the unique map in pullback diagram. Because of universal property of pullback, $f'u = \beta$. In addition, $\alpha' ui = hi = \alpha'\alpha$ which shows both maps $A \xrightarrow{\alpha} X \times_Y Y'$ and $A \xrightarrow{i} B \xrightarrow{u} X \times_Y Y'$ fit into the same pullback diagram. Therefore, the universal property of pullback shows $ui = \alpha$. Hence u is a solution for the original lifting problem.

□

Lemma 3.4. *If* f *and* g *are maps in* $\mathcal{DGM}(R)$ *such that* gf *is defined and if two of three maps* f, g *and* gf *are quasi-isomorphisms, then so is the third one.*

Proof. As $H_*(gf) = H_*(g)H_*(f)$, if two of three of these maps are isomorphisms, then so is the third one. □

Lemma 3.5. *Let R be a differential graded ring and S^n be nth suspension of R. In addition suppose, D^n denotes the mapping cone of the identity map of S^{n-1}. If X is a DG-module, then the following hold.*

i. $Mor_R(S^{n-1}, X) \cong Z_{n-1}X$.

ii. $Mor_R(D^n, X) \cong X_n$.

iii. *The sequence* $0 \longrightarrow Mor_R(S^n, X) \longrightarrow Mor_R(D^n, X) \longrightarrow Mor_R(S^{n-1}, X) \longrightarrow 0$ *is short exact, if* $H_*(X)$=0.

Proof. (i)By 2.17 we have

$$Mor_R(S^{n-1}, X) \cong Z_{n-1}Hom_R(R, X) \cong Z_{n-1}X$$

(ii) Combining 2.17 and 2.40 yields

$$Mor_R(D^n, X) \cong Z_0Hom(D^n, X) \cong Z_0cone(1_{\Sigma^n X}) \cong Kerd_0$$

where $d_0 = \begin{pmatrix} -d^X_{n-1} & 0 \\ 1_{X_{n-1}} & d^X_n \end{pmatrix} : X_{n-1} \oplus X_n \longrightarrow X_{n-2} \oplus X_{n-1}$ and hence $Kerd_0 \cong X_n$.

(iii) Finally, the short exact sequence $0 \longrightarrow S^{n-1} \longrightarrow D^n \longrightarrow S^n \longrightarrow 0$ splits linearly therefore after applying the functor $Hom_R(-, X)$ the resulting sequence is short exact as well. In addition, $H_*(X) \cong 0$ and also S^{n-1} and D^n are semi-projective modules. Thus,

$$Hom_R(S^{n-1}, X) \simeq Hom_R(D^n, X) \simeq 0$$

meaning that the composition of the functors $Z(-)$ and $Hom_R(-, X)$ is exact here, follows from 2.9. $\square$

Lemma 3.6. *A map* $f : X \longrightarrow Y$ *in* $\mathcal{DGM}(R)$ *is surjective if and only if it has the right lifting property with respect to* $i : 0 \longrightarrow D^n$ *for all* $n \in \mathbb{Z}$.

Proof. First suppose f has the lifting property. For an arbitrary $y \in Y$, there is a $n \in \mathbb{Z}$ such that $y \in Y_n$. As a result of the lifting property, there exists the diagram

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "\tilde{y}"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

where $\tilde{y}\begin{pmatrix}1\\0\end{pmatrix} = y$, $\tilde{y}\begin{pmatrix}0\\1\end{pmatrix} = dy$ and $fh = \tilde{y}$. Hence, $f(h\begin{pmatrix}1\\0\end{pmatrix}) = \tilde{y}\begin{pmatrix}1\\0\end{pmatrix} = y$ which shows f is surjective. Conversely, by considering that D^n is a projective object, a lift always exists as f is a surjection. $\square$

3.2 Projective Model on DG-Modules

In [19], it has been shown that the category of complexes over a ring is a cofibrantly generated model category. In the following section, we prove a similar result in $\mathcal{DGM}(R)$ and the strategy is almost analogous to [19] but some modifications are needed. To be more precise, we employ Theorem 1.36 rather than verifying the axioms directly. However finding the sets of generators is the challenging part.

Definition 3.7. Let I and J be sets of the maps $I = \{\, S^{n-1} \longrightarrow D^n \,\}$ and $J = \{\, 0 \longrightarrow D^n \,\}$ where $n \in \mathbb{Z}$. Define a map to be a *fibration* if it is in J–inj and to be a *cofibration* if it is in I–cof. In addition, consider the class of quasi-isomorphisms as the class of weak equivalences and denote it by $\mathcal{W}$.

Proposition 3.8. *A map $p : X \longrightarrow Y$ in $\mathcal{DGM}(R)$ is a fibration if and only if it is surjective.*

Proof. It is a rephrasing of 3.6. $\square$

The following proposition is analogous to [19] and we apply the same idea but results from the second chapter play prominent roles.

Proposition 3.9. *A map $p : X \longrightarrow Y$ in $\mathcal{DGM}(R)$ is a trivial fibration if and only if it is a member of I–inj; i.e I–$inj = \mathcal{W} \cap J$–inj.*

Proof. By 3.5, the set of commutative diagrams below

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D^n \arrow[r, "g"] & Y \end{tikzcd}$$

is in one to one correspondence with $T = \{(y, x) \in Y_n \oplus Z_{n-1}X | p(x) = \delta_n(y)\}$ where δ is the module differential and $Z_{n-1}X$ is cycles of X_{n-1}. Additionally, a lift is $z \in X_n$ such that $\delta(z) = x$ and $p(z) = y$.

Suppose $p \in I\text{–}inj$, we claim that p is a surjective quasi-isomorphism. First of all $Z_n(p) : Z_n(X) \longrightarrow Z_n(Y)$ is surjective, because if $y \in Z_n(Y)$ then the pair $(y, 0)$ belongs to the set T. Due to existence of the lift, there is $z \in X_n$ such that $p(z) = y$ and $\delta(z) = 0$ so $z \in Z_n(X)$ and therefore $Z_n(p)$ is surjective. Furthermore, it shows that $H_n(p) : H_n(X) \longrightarrow H_n(Y)$ is a surjective map.

In this stage, we try to show that $H_n(p) : H_n(X) \longrightarrow H_n(Y)$ is injective. Let $x \in Z_n(X)$, $\bar{x}$ be its homology class and $H_n(p)(\bar{x}) = 0$ so $p(x) \in B_nY$. Hence, $p(x) = \delta(y)$ for some $y \in Y_{n+1}$ and therefore (y, x) belongs to the set T so there is a $z \in X_{n+1}$ such that $\delta z = x$. Thus, $\bar{x} = 0$ and therefore $H_n(p)$ is injective and therefore p is a quasi-isomorphism and as a result by 2.9 p is surjective.

Conversely, suppose p is a trivial fibration (surjective quasi-isomorphism) and K denotes the kernel of p. Applying the functors $Mor_R(D^n, -)$ and $Mor_R(S^{n-1}, -)$ on the exact sequence $0 \longrightarrow K \xrightarrow{j} X \xrightarrow{p} Y \longrightarrow 0$ yields the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Mor(D^n, K) \arrow[r, "j_D"] \arrow[d, "i_K"] & Mor(D^n, X) \arrow[r, "p_D"] \arrow[d, "i_X"] & Mor(D^n, Y) \arrow[r] \arrow[d, "i_Y"] & 0 \\ 0 \arrow[r] & Mor(S^{n-1}, K) \arrow[r, "j_S"] & Mor(S^{n-1}, X) \arrow[r, "p_S"] & Mor(S^{n-1}, Y) \arrow[r] & 0 \end{tikzcd}$$

in which i, j and p induce their related maps and rows are exact, because $H_*(K)$=0 and both D^n and S^{n-1} are semi-projective objects. Moreover, i_K is surjective by 3.5 and by assumption $p_S(f)$=$i_Y(g)$. For $g \in Mor(D^n, Y)$ there exists $h_0 \in Mor(D^n, X)$

such that $p_D(h_0)=g$. Now, by diagram chase, we build h such that $p_D(h)=g$ and $i_X(h)=f$.

By assumptions, $i_X(h_0) - f \in Kerp_S=Img(j_S)$ which means there exists $h_1 \in Mor(S^{n-1}, K)$ such that $j_S(h_1)=i_X(h_0) - f$. As i_K is a surjection there exists $h_2 \in Mor(D^n, K)$ such that $i_K(h_2)=h_1$. Define $h := h_0 - j_D(h_2)$, then

$$\begin{aligned} p_D(h) &= p_D(h_0) - p_D(j_D(h_2)) \\ &= g \qquad \text{as} \qquad p_D j_D = 0. \end{aligned}$$

Furthermore, $i_X(h)=i_X(h_0) - j_S i_K(h_2)=f$. Thus, h is a lift for the first diagram.

□

Proposition 3.10. *With the above notations, J–cell $\subseteq \mathcal{W} \cap I$–cof.*

Proof. By Lemma 1.32 *J–cell* $\subseteq$ *J–cof*. Combining 3.2 and 3.10 shows that *I–inj* $\subseteq$ *J–inj* and therefore *J–cof* $\subseteq$ *I–cof*. Thus, *J–cell* $\subseteq$ *I–cof*.

Now, it suffices to show that *J–cell* $\subseteq \mathcal{W}$. Let L be the class of all injective quasi-isomorphisms so $J \subseteq L$. By 3.1 and 2.26 L is closed under pushout and transfinite composition. Hence *J–cell* $\subseteq L \subseteq \mathcal{W}$.

□

Lemma 3.11. *Every trivial cofibration is an injective map with a projective cokernel.*

Proof. Let $f : A \longrightarrow B$ be a quasi-isomorphism in *I–cof* = (*I–inj*)*–proj*. Thus, it has the left lifting property with respect to all surjective quasi-isomorphisms then by dual of 3.3 $Cokerf$ has the left lifting property with respect to all surjective quasi-isomorphisms and therefore it is a semi-projective module.

In next step, we claim that f is injective and therefore as it is a quasi-isomorphism we can conclude that $H_*(Cokerf)=0$. Because cokernel of f is both quasi-trivial and semi-projective it follows from 2.62 that it is a projective object.

Let $D(A)$ be the mapping cone of the identity map of A. Then, there is a natural injective map $i : A \longrightarrow D(A)$. Additionally, $0 : D(A) \longrightarrow 0$ is a surjective quasi-isomorphism. Hence there is the commutative diagram below in which the map h is a lift.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & D(A) \arrow[d, "0"] \\ B \arrow[r, "0"'] \arrow[ur, dotted, "h"] & 0 \end{tikzcd}$$

For a given $x \in Kerf$ then $0 = hf(x) = i(x)$ and because i is injective, $x = 0$ and therefore f is injective.

$\square$

Proposition 3.12. *The class of trivial cofibrations is a subclass of J–cof. In other words, $\mathcal{W} \cap I$–$cof \subseteq J$–cof.*

Proof. Let $f : A \longrightarrow B$ be a trivial cofibration with cokernel C and $p : X \longrightarrow Y$ be an arbitrary member of J–inj. Then, any lifting problem can be considered as

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i_2"'] & X \arrow[d, "p"] \\ C \oplus A \arrow[r, "\beta=(\beta_1,\beta_2)"] \arrow[ur, dotted, "h=(h_1,h_2)"] & Y \end{tikzcd}$$

First of all, as C is projective object and p is a surjection there exists a map h_1 such that $ph_1{=}\beta_1$ in addition, define $h_2 := \alpha$. In the diagram h is a lift, because $hi_2{=}h_2{=}\alpha$ and $ph{=}(ph_1, p\alpha){=}(\beta_1, \beta_2)$. Therefore f has the left lifting property with respect to J–inj and hence it is a member of J–cof.

$\square$

Theorem 3.13. *The category $\mathcal{DGM}(R)$ is a model category by letting J–inj be the class of fibrations, I–cof be the class of cofibrations and the quasi-isomorphisms be the class of weak equivalences.*

Proof. We verify the conditions of Theorem 1.36. First of all small limits and colimits exist in $\mathcal{DGM}(R)$ because arbitrary products and coproducts as well as pullbacks and pushouts exist in $\mathcal{DGM}(R)$. For condition (i) it is clear that $\mathcal{W}$ is a subcategory and has the two of three property by 3.4. To show the condition (ii) and (iii) note that $\mathcal{DGM}(R)$ is a Grothendieck category and therefore by [20, 1.2] every object is small. The proposition 3.10 is a proof for condition (iv) and finally, 3.2 and 3.12 verify the remaining conditions. □

3.3 More on Cofibration in The Projective Model

Lemma 3.11 describes the class of trivial cofibrations for the projective model structure but does not provide necessary and sufficient conditions. However an explicit description of cofibrations is given in the rest of this section.

Lemma 3.14. *Let K be an element of $\mathcal{DGM}(R)$ and $H_*(K) = 0$, if P is a semi-projective module in $\mathcal{DGM}(R)$ then every map $f : P \longrightarrow K$ is null homotopic.*

Proof. It follows from [4, 9.6.1] that $H_*(Hom(P, K)) = 0$ if $H_*(K) = 0$. Considering the fact that $H_0(Hom_R(P, K)) = Mor_R(P, K)/\sim$ (modulo homotopy) the map f becomes homotopic to zero. □

Although the idea of the following theorem is analogous to [19, 2.3.9], applying the idea of the mapping cone plays the main role.

Theorem 3.15. *Suppose the projective model structure has been defined on $\mathcal{DGM}(R)$, then a map $i : A \longrightarrow B$ is a cofibration if and only if it is injective with semi-projective cokernel and linearly split.*

Proof. Let $i : A \longrightarrow B$ be a cofibration so it has the left lifting property with respect to all trivial fibrations. In the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, hook, "i"'] & Cone(1_A) \arrow[d, two heads, "\cong"] \\ B \arrow[r, "0"'] \arrow[ur, dotted, "h"] & 0 \end{tikzcd}$$

the lift h exists. Because j is an injective map and $hi = j$ so i is an injective map. In addition any pushout of i is a cofibration hence $0 : 0 \longrightarrow Coker(i)$ is a cofibration which means that $Coker(i)$ is a semi-projective module. Furthermore, considering the fact that $Coker(i)$ is a linearly projective module the following exact sequence splits linearly.

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow Coker(i) \longrightarrow 0$$

Suppose $i : A \longrightarrow B$ is an injective map with semi-projective cokernel and C denotes the $Coker(i)$. A given lifting problem can be considered as the diagram

$$\begin{tikzcd}
 & K \arrow[d, "j"] \\
A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "p", "\simeq"'] \\
B \approx A \oplus C \arrow[r, "\beta"'] \arrow[d] & Y \\
C &
\end{tikzcd} \tag{3.3.1}$$

where B is linearly isomorphic to $A \oplus C$, p is a trivial fibration, $K{=}Ker(p)$ and $H_*(K){=}0$. In addition $d^B(a,c) = (d^A a + \tau c, d^C c)$ where $\tau : C \longrightarrow \Sigma A$ is a map such that $d^A\tau + \tau d^C = 0$. The map $\beta : (A \oplus C, d^B) \longrightarrow Y$ must be defined such that $\beta(a,c) = \beta(a,0) + \beta(0,c) = \beta i a + \sigma c = p\alpha a + \sigma c$ where $\sigma : C \longrightarrow Y$ is a map such that $d^Y\sigma = p\alpha\tau + \sigma d^C$ due to the fact that β is a chain map. In the above diagram, a lift is equal to a map $h = (h_1, h_2)$ such that $h_1 = \alpha$ and $h_2 = \nu$ where $p\nu = \sigma$ and $d^X\nu = \alpha\tau + \nu d^C$ and therefore a lift is equivalent to the map $\nu : C \longrightarrow X$ with mentioned properties.

Considering that C is a semi-projective module, there is a chain map $G : C \longrightarrow X$ such that $pG = \sigma$ but $d^X G = \alpha\tau + G d^C$ is not always true. Let

$$r = d^X G - G d^C - \alpha\tau : C \longrightarrow \Sigma X$$

be a map of graded modules then

$$d^X r = (d^X)^2 G - d^X G d^C - d^X \alpha\tau = -d^X G d^C - \alpha d^A \tau = -d^X G d^C + \alpha\tau d^C = -r d^C$$

Hence $r d^C = d^{\Sigma A} r$ meaning that r is a morphism of DG modules.

Due to the fact that $0 = pr = pd^X G - \sigma d^C - d^Y \sigma + \sigma d^C$ we can factor r via $Ker(p)$ and we have $C \xrightarrow{s} \Sigma K \xrightarrow{j} \Sigma X$ additionally by 3.14

$$s = d^{\Sigma K} D + D d^C = -d^K D + D d^C$$

Now, let $\nu = G + jD$ then firstly, $p\nu = PG + PjD = \sigma$ and secondly, $d^X \nu = d^X G + j(-s + Dd^C) = r + Gd^C + \alpha\tau - r + jDd^C = \alpha\tau + (G + jD)d^C = \alpha\tau + \nu d^C$, hence $h = (\alpha, G + jD)$ is our desired lift. $\square$

3.4 More on Semi-injective DG-Modules

Recall that for an ordinary ring K, a module is injective if it has the right lifting property with respect to all injective maps. It is quite difficult to verify this condition with respect to the whole class of injective maps which is even not a set. However Baer's criterion provides a useful tool for checking whether a module is injective. In fact, the criterion states that checking the lifting problem for the set of all inclusion from ideals of K is enough. For all sorts of injectivity [1] and [33] may be good resources to start.

The main aim of this section is to find a certain set of morphisms in $\mathcal{DGM}(R)$ such that to verify semi-injectivity of a module, considering just this certain set instead of the whole class of injective quasi-isomorphisms would be sufficient. For the rest of this section, $|C|$ denotes the cardinality of the set C.

Lemma 3.16. *Suppose $i : A \longrightarrow B$ is an injective quasi-isomorphism in $\mathcal{DGM}(R)$. For every submodule C of B in $\mathcal{DGM}(R)$ with $|C| \leq \gamma$ where $\gamma = \aleph_0 + |R|$, there is a submodule D of B in $\mathcal{DGM}(R)$ containing C such that $|D| \leq \gamma$ and $i : A \cap D \longrightarrow D$ is a weak equivalence.*

Proof. Suppose $i : A \cap C \longrightarrow C$ is not a weak equivalence then $H_*(C/A \cap C) \neq 0$. For every $\alpha \in H_*(C/A \cap C)$ choose $z_\alpha \in C$ such that the homology class of its quotient class is equal to α. Since $H_*(B/A) = 0$, there is $b_\alpha \in B$ such that $\delta(b_\alpha) - z_\alpha \in A$. Let C_1 be the smallest DG submodule of B which contains C and

$$L = \{z_\alpha, b_\alpha | \alpha \in H_*(C/A \cap C)\}.$$

Hence the induced map in homology $H_*(C/A\cap C) \longrightarrow H_*(C_1/A\cap C_1)$ is a zero map and $|C_1| \leq |L||R| \leq \gamma$. Iterating this construction forms a sequence C_n of modules. Let D be the colimit of the sequence C_n with the inclusion as the map between its elements. For the cardinality of D we have $|D| \leq \aleph_0 |C_n| \leq \gamma$. Consider the direct family of exact sequences,

$$0 \longrightarrow C_n \cap A \longrightarrow C_n \longrightarrow C_n/C_n \cap A \longrightarrow 0$$

then we have $\varinjlim (C_n/C_n \cap A) \cong D/D\cap A$ and therefore

$$H_*(D/A\cap D) = \varinjlim H_*(C_n/C_n\cap A) \cong 0.$$

Because the direct limits preserve exact sequences by Proposition 2.35 and commute with homology by Theorem 2.36. Therefore $i : D\cap A \longrightarrow D$ is an injective weak equivalence. $\square$

An analogy to the above lemma has been proved for the category of chain complexes for a Grothendieck category; cf. [20, 2.10].

Lemma 3.17. *Suppose J is a set containing a map for each isomorphism class of injective quasi-isomorphisms $j : M \hookrightarrow N$ with $|N| \leq \gamma$ for $\gamma = \aleph_0 + |R|$. For a given injective quasi-isomorphism $i : A \hookrightarrow B$ the object B has a filtration $\{B_\alpha\}$, such that the map $i_\alpha : A \hookrightarrow B_\alpha$ is an injective quasi-isomorphism and the embedding $j_{\alpha,\alpha+1} : B_\alpha \hookrightarrow B_{\alpha+1}$ is the pushout along a member of J.*

Proof. We construct the filtration. Let $B_1 := i(A)$ and suppose ζ is an ordinal with associated cardinal equal to $|B|$ and assume for an ordinal $\alpha < \zeta$, B_α has been constructed. Let $x_{\alpha+1} \in B \setminus B_\alpha$ then by using Lemma 3.16 for $j_\alpha : B_\alpha \hookrightarrow B$, there exists an object $D_{\alpha+1}$ containing $x_{\alpha+1}$ such that $j_{x_{\alpha+1}} : B_\alpha \cap D_{\alpha+1} \hookrightarrow D_{\alpha+1}$ is a member of J. By considering the pushout diagram

$$\begin{tikzcd}
B_\alpha \cap D_{\alpha+1} \arrow[r, hook] \arrow[d, hook, "j_{x_{\alpha+1}}"'] & B_\alpha \arrow[d, "j_{\alpha,\alpha+1}"] \\
D_{\alpha+1} \arrow[r] & B_\alpha + D_{\alpha+1}
\end{tikzcd}$$

define $B_{\alpha+1} := B_\alpha + D_{\alpha+1}$ and $j_{\alpha+1} := j_{\alpha,\alpha+1} \circ j_\alpha$. Furthermore, as $j_{\alpha,\alpha+1}$ is the pushout of an injective quasi-isomorphism, it is also an injective quasi-isomorphism. For the limit ordinal $\beta < \zeta$ define $B_\beta := \varinjlim_{\alpha<\beta}(B_\alpha)$. As the map $i_\beta : A \hookrightarrow B_\beta$ is the transfinite composition of injective quasi-isomorphisms, then it is an injective quasi-isomorphism. Finally by construction B=$\varinjlim_\alpha(B_\alpha)$. □

Remark 3.18. Every quasi-trivial object B is the colimit of a direct family B_α such that $B_\alpha \longrightarrow B_{\alpha+1}$ is an embedding and $H_*(B_\alpha)$=0 for all α and $|B_\alpha| \leq \gamma$ for $\alpha \leq \omega$ where ω is the first limit ordinal.

Proposition 3.19. *An object $I \in \mathcal{DGM}(R)$ is semi-injective if and only if it is a member of J–inj.*

Proof. The necessity of the proposition is obvious. For sufficiency of the proposition, assume $i : A \longrightarrow B$ is an arbitrary injective quasi-isomorphism and $f : A \longrightarrow I$ is an arbitrary map. By using the introduced filtration in Lemma 3.17 we build our desired morphism. Suppose there exists a map $u_\alpha : B_\alpha \longrightarrow I$ such that f=$u_\alpha i_\alpha$. In the diagram

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "i_\alpha"'] & I \\
B_\alpha \cap D_{\alpha+1} \arrow[r] \arrow[d, "j_x"'] & B_\alpha \arrow[ur, "u_\alpha"] \arrow[d, "i_{\alpha,\alpha+1}"'] & \\
D_{\alpha+1} \arrow[r] \arrow[uurr, bend right, "h_{\alpha+1}"] & B_{\alpha+1} \arrow[uur, bend right, "u_{\alpha+1}"'] &
\end{tikzcd}$$

the map $i_{\alpha,\alpha+1}$ is the pushout of j_x and $h_{\alpha+1}$ exists because j_x belongs to J. Additionally, $u_{\alpha+1}$ exists due to the uniqueness of pushout and $u_{\alpha+1}i_{\alpha+1}$=$u_\alpha i_\alpha$=f where $i_{\alpha+1}$=$i_{\alpha,\alpha+1}i_\alpha$. Now $u : \varinjlim B_\alpha \xrightarrow{\varinjlim u_\alpha} I$ is a map such that ui=f so it is our desired lift. □

3.5 Injective Model on DG-Modules

The main aim of this section is to define a model structure on $\mathcal{DGM}(R)$ by considering injective maps as the class of cofibrations. Through this section we try to prove Theorem 3.22 with a strategy based on showing that all the axioms of the definition 1.2 hold.

Definition 3.20. Define a map $f \in \mathcal{DGM}(R)$ to be a *weak equivalence* if it is a quasi-isomorphism and let $\mathcal{W}$ be the class of all weak equivalences. Define a map to be a *cofibration* if it is an injective map and define a map to be a *fibration* if it has right lifting property with respect to all injective quasi-isomorphisms.

Remark 3.21. Recall that a map is a trivial cofibration if it is a cofibration and weak equivalence and a map is a trivial fibration if it is a fibration and weak equivalence.

Theorem 3.22. *Definition 3.20 gives a model structure on $\mathcal{DGM}(R)$ called the injective model.*

In the rest of this section, we provide a proof for Theorem 3.22.

Lemma 3.23. *The classes of weak equivalences, cofibrations and fibrations are closed under composition and contain identity maps.*

Proof. **Weak equivalence** For maps f and g, we have $H_*(gf) = H_*(g)H_*(f)$ and therefore the composition of weak equivalences is a quasi-isomorphism. Hence the class of weak equivalences is closed under composition. In addition, identity maps are weak equivalences.

Cofibration Let f and g be injective maps. Then gf is injective so the class of cofibration is closed under composition. Furthermore, identity maps are injective and therefore belong to the class of cofibrations.

Fibration Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two maps which have the right lifting property with respect to all trivial cofibrations. Consider the following arbitrary commutative diagram such that $i : A \longrightarrow B$ is a trivial cofibration.

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "gf"] \\
B \arrow[r, "\beta"'] & Z
\end{tikzcd}
\tag{3.5.1}
$$

Breaking the diagram 3.5.1 yields the diagram 3.5.2 in which h exists due to the lifting property of g.

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[r, "f"] & Y \arrow[d, "g"] \\
B \arrow[ur, dotted, "h'"] \arrow[urr, dotted, "h"'] \arrow[rr, "\beta"'] & & Z
\end{tikzcd}
\tag{3.5.2}
$$

By considering the upper triangle of the diagram 3.5.2 we have the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
B \arrow[ur, dotted, "h'"] \arrow[r, "h"'] & Y
\end{tikzcd}
$$

where h' exists because of lifting property of f. Additionally, $fh' = h, h'i = \alpha$ and therefore $gfh' = gh = \beta$. Hence, h' is a lift for diagram 3.5.1. □

Corollary 3.24. *The first and second axiom of model categories (**MC1** and **MC2**) hold in $\mathcal{DGM}(R)$.*

Proposition 3.25. *Let f be retract of g then*

i. If g is a fibration then so is f.

ii. If g is a cofibration then so is f.

iii. If g is a weak equivalence then so is f.

Proof. **Fibration** Since g is a fibration it has right lifting property with respect to all trivial cofibrations. Consider an arbitrary commutative diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "i"'] & A \arrow[d, "f"] \\ Y \arrow[r, "\beta"'] & B \end{tikzcd} \tag{3.5.3}$$

when i is a trivial fibration. We must show there is a lift in the above diagram. Since f is retract of g we can extend the diagram to following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "j"'] & A \arrow[r, "i"] \arrow[d, "f"] & A' \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\ Y \arrow[r, "\beta"'] \arrow[urr, dotted, "h"] & B \arrow[r, "i'"'] & B' \arrow[r, "r'"'] & B \end{tikzcd}$$

because g is a fibration there is a map $h : Y \longrightarrow A'$ such that $jh = i\alpha$ and $gh = i'\beta$. The map $rh : Y \longrightarrow A$ is a lift for the diagram 3.5.3 and $rhj = ri\alpha = \alpha$ also $frh = r'gh = r'i'\beta = \beta$

Cofibration Let g be a cofibration, consider the first left square of the retract diagram as below

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ B \arrow[r, "i'"'] & B' \end{tikzcd}$$

then

$$f(x_1) = f(x_2) \Longrightarrow i'f(x_1) = i'f(x_2) \Longrightarrow gi(x_1) = gi(x_2)$$
$$\Longrightarrow i(x_1) = i(x_2) \Longrightarrow ri(x_1) = ri(x_2) \Longrightarrow x_1 = x_2$$

Weak equivalence Let g be a weak equivalence. Consider the homology of the retract diagram

$$\begin{tikzcd}
H_*(A) \arrow[r, "H_*(i)"] \arrow[d, "H_*(f)"] & H_*(A') \arrow[r, "H_*(r)"] \arrow[d, "H_*(g)"] & H_*(A) \arrow[d, "H_*(f)"] \\
H_*(B) \arrow[r, "H_*(i')"'] & H_*(B') \arrow[r, "H_*(r')"'] & H_*(B).
\end{tikzcd}$$

As $H_*(g)$ is an isomorphism, it is both surjective and injective. The left square shows $H_*(f)$ is injective, because $H_*(g)$ is injective (same calculation as cofibration part). Moreover, right square shows $H_*(f)$ is surjective due to the fact that $H_*(g)$ is a surjection. Hence $H_*(f)$ is an isomorphism and therefore f is a weak equivalence. □

Corollary 3.26. *The retract axiom (**MC3**) holds in $\mathcal{DGM}(R)$.*

Proposition 3.27. *Every fibration $p : X \longrightarrow Y$ is a surjection with fibrant, i.e,semi-injective kernel. In addition, every trivial fibration is surjective with injective kernel and has a right inverse.*

Proof. **Fibration** Since a fibration has the left lifting property with respect to all injective and quasi-isomorphism maps, in particular it has left lifting property with respect to $\{ f : 0 \longrightarrow D^n | n \in \mathbb{Z} \}$ therefore by 3.6 it is surjective. As $0 : kerp \longrightarrow 0$ is a pullback of p then by 3.3 it has the left lifting property with respect to all trivial cofibrations so $kerp$ is a fibrant object and by 2.75 it is a semi-injective module.

Trivial fibration For a fibration p the exact sequence

$$0 \longrightarrow Kerp \longrightarrow X \xrightarrow{p} Y \longrightarrow 0$$

exists. If p is a trivial fibration, $H_*(Kerp) = 0$ and therefore $Kerp$ is a quasi-trivial semi-injective module. Thus, by 2.77 it is an injective object.

□

Proposition 3.28. *Trivial fibrations have the right lifting property with respect to all cofibrations (injections).*

Proof. A trivial fibration is surjective with injective kernel by the previous proposition. In addition, by 3.2 any surjection with injective kernel has the right lifting property with respect to all injections. Hence a trivial fibration has the right lifting property with respect to all cofibrations. $\square$

Corollary 3.29. *The lifting axiom (**MC4**) holds in $\mathcal{DGM}(R)$.*

Proof. Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

If i is a trivial cofibration and p is a fibration then by definition a lift exists. If i is a cofibration and p is a trivial fibration then by Proposition 3.28 a lift exists. $\square$

Remark 3.30. Let I be the class of injective maps, I' be the class injective quasi-isomorphisms and $\mathcal{W}$ be the class of weak equivalences. Then I'–inj is the class of fibrations and the following holds.

i. $\mathcal{W} \cap I'\text{–}inj \subseteq$ {Surjective maps with injective kernel} (By 3.27)

ii. $I' \subseteq I \Longrightarrow I\text{–}inj \subseteq I'\text{–}inj$

iii. {Surjective maps with injective kernel } $\subseteq \mathcal{W}$ (Injective objects are acyclic by 2.77)

iv. {Surjective maps with injective kernel } $\subseteq I\text{–}inj \subseteq I'\text{–}inj$ (By 3.2)

v. {Surjective maps with injective kernel } $\subseteq I'\text{–}inj \cap \mathcal{W}$ (iii, iv)

vi. {Surjective maps with injective kernel } $= I'\text{–}inj \cap \mathcal{W} =${Trivial fibration} (i,v)

Proposition 3.31. *A map $p : X \longrightarrow Y$ in $\mathcal{DGM}(R)$ is a fibration if and only if it is a surjection with semi-injective kernel.*

Proof. The necessary condition has been proved earlier. For the sufficient condition suppose we have an arbitrary commutative square below in which i is a trivial cofibration (injective quasi-isomorphism).

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

If K denotes the semi-injective kernel of p then there is a linearly split short exact sequence $0 \longrightarrow K \longrightarrow X \xrightarrow{p} Y \longrightarrow 0$. Hence $(X, d^X) \cong (K^\sharp \oplus Y^\sharp, \begin{pmatrix} d^K & \tau \\ 0 & d^Y \end{pmatrix})$ such that $\tau : Y \longrightarrow \Sigma K$ is a map with property that $d^K\tau + \tau d^K = 0$. In addition $\alpha = \begin{pmatrix} \alpha_1 \\ \beta i \end{pmatrix}$ and as it is a chain map then $\alpha_1 d^A = d^K\alpha_1 + \tau\beta i$.

In the diagram a lift h is equal to $\begin{pmatrix} h_1 \\ \beta \end{pmatrix}$ where $h_1 : B \longrightarrow K$, and $h_1 i = \alpha_1$ and $h_1 d^B = d^K h_1 + \tau\beta$ because h is a chain map and commutes with d^X. To find h_1 we use different properties of a semi-injective modules. Considering that K is a semi-injective module then there is a map $g : B^\sharp \longrightarrow K^\sharp$ such that $gi = \alpha_1$. The map $g_1 {=} gd^B - d^K g - \tau\beta : B \longrightarrow \Sigma K$ is in fact a chain map of $Mor(B, \Sigma K)$. As ΣK is semi-injective and $H_*(B/A){=}0$, applying the functor $Mor(-, \Sigma K)$ on the exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} B/A \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow Mor(B/A, \Sigma K) \xrightarrow{\bar{q}} Mor(B, \Sigma K) \xrightarrow{\bar{i}} Mor(A, \Sigma K) \longrightarrow 0 \,.$$

Additionally, we have

$$\begin{aligned} \bar{i}(g_1) &{=} gd^B i - d^K gi - \tau\beta i \\ &{=} \alpha_1 d^A - d^K\alpha_1 - \tau\beta i {=} 0 \end{aligned}$$

which means $g_1 \in Ker\,\bar{i}$ and therefore there is a g_2 such that $\bar{q}(g_2) = g_1$. Since ΣK is a semi-injective module and B/A is a quasi-trivial, every map in $Mor(B/A, \Sigma K)$ is null homotopic. Hence, there is a homotopy $H : B/A \longrightarrow K$ such that

$$Hd^{B/A} + d^{\Sigma K} H {=} g_2 = Hd^{B/A} - d^K H.$$

Now we can define h_1, in fact $h_1 := g - \bar{q}(H)$, we claim h_1 has the desired properties. First of all, $\bar{i}(h_1)$=$gi = \alpha_1$ and also

$$\begin{aligned}(g-\bar{q}(H))d^B = d^K(g-\bar{q}(H)) + \tau\beta &\leftrightarrow gd^B - d^kg - \tau\beta = Hqd^B - d^KHq\\ &\leftrightarrow gd^B - d^kg - \tau\beta = Hd^{B/A}q - d^KHq\\ &\leftrightarrow gd^B - d^kg - \tau\beta = \bar{q}(g_2) = g_1.\end{aligned}$$

□

In the rest of this section, a proof for the factorization axiom is provided. The next proposition gives a generalization of Baer's criterion in a Grothendieck category and its proof can be found in [23, 8.4.7].

Proposition 3.32. *Let $\mathcal{C}$ denote a Grothendieck category and let $\{G_i\}_{i\in I}$ be a system of generators. Then an object $z \in \mathcal{C}$ is injective if and only if for any $i \in I$ and any subobject $w \subset G_i$, the natural map $Mor(G_i, z) \longrightarrow Mor(w, z)$ is surjective.*

Proposition 3.33. *Every morphism $f : X \longrightarrow Y$ in $\mathcal{DGM}(R)$ can be factored to $f = pi$ such that i is a cofibration and p is a trivial fibration.*

Proof. Let $L = \{\ j : w \hookrightarrow D^n\ \}$ with w a submodule of D^n and $n \in \mathbb{Z}$. By small object argument we can factor f as $f = pi$ such that $i \in L$–*cell* and $p \in L$–*inj*. We will show that i is injective and p is a surjective quasi-isomorphism with injective kernel. At first we discuss the properties of p.

surjection As p has the left lifting property with respect to all elements of L so by 3.6 it is surjective.

injective ker Because p has left lifting property with respect to L then by 3.3 $Kerp$ has lifting property as well so by Proposition 3.33 it is an injective object.

quasi-iso In addition $KerP$ would be acyclic because it is injective hence p would be a quasi-isomorphism.

Regarding i, let I be the class of all injective maps then $L \subset I$ and therefore L–*cell* $\subset I$. Because, by 3.1 and 2.26 I is closed under pushout and transfinite compositions.

□

The proof of the next proposition is an adaptation of [19, 2.3.5].

Proposition 3.34. *In category $\mathcal{DGM}(R)$, consider the set*

$$J = \{\, i : X \longrightarrow Y \ \mid |Y| \leq \gamma\}$$

in which γ=$\aleph_0 + |R|$ and containing just one element of each isomorphism class of injective quasi-isomorphisms. Then the class of injective quasi-isomorphisms I' is a subclass of J–cof; $I' \subseteq J$–cof.

Proof. Suppose $i : A \longrightarrow B$ is an injective quasi-isomorphism and $p : X \longrightarrow Y$ is an arbitrary member of J–inj. We claim that i has the left lifting property with respect to p. Consider a commutative diagram as below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd} \tag{3.5.4}$$

Let T be the set of all partial lifts (C, h) such that $i' : A \longrightarrow C$ is an injective quasi-isomorphism and $i'A \subseteq C \subseteq B$ and $h : C \longrightarrow X$ is a partial lift for the diagram. The pair (iA, fi^{-1}) belongs to T and so it is not empty. Additionally, T is partially ordered and has a upper bound so by Zorn's Lemma a maximal element of T, (M, h) exists. If M is not all of B, choose $x \in B \setminus M$ and denote its generated module by $\langle x \rangle$ then $|\langle x \rangle| \leq \gamma$. By Lemma 3.16 there is a submodule D of B containing x with $|D| \leq \gamma$ such that the inclusion $D \cap A \hookrightarrow D$ is a member of J. In the diagram

$$\begin{tikzcd} D \cap A \arrow[r, hook] \arrow[d, hook] & M \arrow[d, "i_2"] \\ D \arrow[r] & N \end{tikzcd}$$

i_2 is a pushout of an element of J, therefore $i_2 \in J$–cof. By considering the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "i"'] & & B \arrow[d, "p"] \\ M \arrow[urr, "h"] \arrow[r, "i_2"'] & N \arrow[ur, "h_1"'] \arrow[r, "g"'] & Y \end{tikzcd} \tag{3.5.5}$$

we can form a square by the triangle lower than the diagonal of the rectangle, in this square a lift h_1 exists because $i_2 \in J$–cof. Now the pair (N, h_1) is a member of T and it is in contradiction with maximality of (M, h). Hence, $M = B$ and $i \in J$–cof. $\square$

Corollary 3.35. *Every morphism $f : X \longrightarrow Y$ in $\mathcal{DGM}(R)$ can be factored out to $f = pi$ such that i is a trivial cofibration and p is a fibration; i.e the axiom (**MC5**) holds in $\mathcal{DGM}(R)$.*

Proof. **First Argument** Recall that I' is the class of injective quasi-isomorphisms.

First of all, I'–$cof = J$–cof. Because $J \subseteq I'$ yields the relation J–$cof \subseteq I'$–cof and also by the previous proposition I'–$cof \subseteq (J$–$cof)$–cof. In addition we can conclude that I'–$inj = J$–inj because we have $(S$–$cof)$–inj=S–inj for every class S.

By the small object argument morphism $f : X \longrightarrow Y$ in $\mathcal{DGM}(R)$ can be factored out to $f = pi$ such that $i \in J$–$cell$ and $p \in J$–inj so p is in I'–inj and therefore by definition it is a fibration. Furthermore $i \in J$–$cell \subseteq I'$ and therefore it is an injective quasi-isomorphism or equivalently a trivial cofibration.

Second Argument

If $i \in J$–$cell$ then i is injective quasi-isomorphism or equivalently a trivial cofibration. If $p \in J$–inj therefore the map $Ker\ p \longrightarrow 0$ is also in J–inj. Hence by 3.19 $Ker\ p$ is a semi-injective module and therefore by 3.31 it suffices to show that p is a surjective map. Due to the fact that the isomorphic class of maps $0 \longrightarrow D^n$ with $n \in \mathbb{Z}$ belong to J Lemma 3.6 yields the desired result. $\square$

3.6 Projective and Injective Resolution

Definition 3.36. Let M be a module in $\mathcal{DGM}(R)$.

i. A map $\pi : P \longrightarrow M$ is called *projective resolution* of M if π is a surjective map with P a projective DG module.

ii. A map $\pi : P \longrightarrow M$ is called *semi-projective resolution* of M if π is a surjective quasi-isomorphism with P a semi-projective DG module.

iii. A map $i : M \longrightarrow I$ is called *injective resolution* of M if i is an injective map with I an injective DG module.

iv. A map $i : M \longrightarrow I$ is called *semi-injective resolution* of M if i is an injective quasi-isomorphism with I a semi-injective DG module.

Sometimes the condition of surjectivity or injectivity of the maps are not required but in their presence the word *strict resolution* is used. In the rest of this section, in order to obtain the resolutions for a DG module, both constructive and non-constructive approaches are provided.

Theorem 3.37. *Let M be a module in $\mathcal{DGM}(R)$.*

i. M has a semi-projective resolution.

ii. M has a semi-injective resolution.

Remark 3.38. A constructive proof of first part can be found in [4, 8.3.3] and [14, 6.5] and for the second part a constructive proof is provided in [4, 10.3.5].

Non-constructive proof 3.37 (i) Suppose the projective model structure is defined on $\mathcal{DGM}(R)$. The map $0 : 0 \longrightarrow M$ can be factored into $0 = pi$ so there exists a module $\hat{M}$ such that $i : 0 \longrightarrow \hat{M}$ is a cofibration and $p : \hat{M} \longrightarrow M$ is a trivial fibration and therefore $Coker\ i = \hat{M}$ is a semi-projective object and p is a surjective quasi-isomorphism which means p is a semi-projective resolution.

(ii) Suppose the injective model structure is defined on $\mathcal{DGM}(R)$. The map $0 : M \longrightarrow 0$ can be factored into $0 = pi$ so there exists a module $\check{M}$ such that $i : M \longrightarrow \check{M}$ is a trivial cofibration and $p : \check{M} \longrightarrow 0$ is a fibration and therefore $Ker\ p = \check{M}$ is a semi-injective object and i is an injective quasi-isomorphism which means i is a semi-injective resolution.

The non-constructive approach relies on the small object argument and the model structures. However, by considering the constructive approach, we can prove the existence of the resolutions directly. This approach is also useful for computational purposes and is based on results from [4] and [14].

Constructive proof 3.37 i. As every semi-free module is semi-projective we construct a semi-free resolution of M and do it by induction on $u \geq 0$ sequences of inclusion of DG modules $L^{u-1} \subseteq L^u$ and of morphisms $\epsilon^u : L^u \longrightarrow M$ of DG modules with $\epsilon^u|_{L^{u-1}} = \epsilon^{u-1}$.

If $u = 0$, then choose a set of cycles Z^0 in M such that $cls(Z^0)$ generates the graded $H(R)-$module $H(M)$, and let

$$E^0 = \{e_z : |e_z| = |z|\}_{z \in Z^0}$$

be a linearly independent set over $R^\natural$. Direct computations show that the formula

$$(L^0)^\natural = R^\natural E^0 \text{ and } \partial(\sum_{z \in Z^0} r_z e_z) = \sum_{z \in Z^0} \partial(r_z) e_z$$

defines a DG module L^0 over R, and that

$$\epsilon^0(\sum_{z \in Z^0} r_z e_z) = \sum_{z \in Z^0} r_z z$$

defines a morphism $\epsilon^0 : L^0 \longrightarrow M$ in $\mathcal{DGM}(R)$.

If $u \geq 0$ and a DG module L^u and a morphism $\epsilon^u : L^u \longrightarrow M$ has been defined, then choose a set of cycles Z^{u+1} of L^u such that $cls(Z^{u+1})$ generates the graded $H(R)-$module $Ker(H(\epsilon^u))$, and let

$$E^{u+1} = \{e_z : |e_z| = |z| + 1\}_{z \in Z^{u+1}}$$

be a linearly independent set over $R^\natural$. Direct computations show that

$$(L^{u+1})^\natural = R^\natural E^u \bigoplus (L^u)^\natural$$

$$\partial(\sum_{z \in Z^{u+1}} r_z e_z + x) = \sum_{z \in Z^{u+1}} \partial(r_z) e_z + \sum_{z \in Z^{u+1}} (-1)^{|r_z|} r_z z + \partial(x)$$

defines a DG module L^{u+1} over R, and that L^u, identified with its canonical image, is a DG submodule of L^{u+1}. By construction, for each $z \in Z^{u+1}$ there exists an element $y_z \in M$ such that $\epsilon^u(z) = \partial(y_z)$. By a computation one can verify that

$$\epsilon^{u+1}(\sum_{z \in Z^{u+1}} r_z e_z + x) = \sum_{z \in Z^{u+1}} r_z y_z + \epsilon^u(x)$$

defines a morphism $\epsilon^{u+1} : L^{u+1} \longrightarrow M$ of DG modules, such that $\epsilon^{u+1}|_{L^u} = \epsilon^u$.

We set $L = \cup_{u\geq 0} L^u$ and note that $E = \sqcup_{u\geq 0} E^u$ is a semi-basis of L and $\epsilon = \varinjlim \epsilon^u : L \longrightarrow M$ is a morphism of DG modules. Now we show that ϵ is a quasi-isomorphism and it is surjective map if and only if Z^0 generates the graded $Z(R)-$module $Z(M)$. For each $u \geq 0$ we have a commutative diagram

$$\begin{tikzcd}
H(L^0) \arrow[r] \arrow[dr, "H(\epsilon^0)"'] & H(L^u) \arrow[r] \arrow[d, "H(\epsilon^u)"] & H(L) \arrow[dl, "H(\epsilon)"] \\
& H(M) &
\end{tikzcd}$$

By construction, $H(\epsilon^0)$ is surjective and therefore $H(\epsilon^u)$ is surjective as well, hence $H(\epsilon)$ is surjective. To show it is injective, consider a cycle $x \in L$ with $H(\epsilon)(cls(x)) = 0$. Considering the fact that $H(\epsilon) = H(\varinjlim_u \epsilon^u) = \varinjlim_u H(\epsilon^u)$ there exists a u such that $x \in L^u$ and $cls(x) \in Ker(H(\epsilon^u))$. By the choice of Z^{u+1}, we have

$$x = \sum_{z\in Z^{u+1}}^{m} a_z z + \partial(y) \in L^u$$

for appropriate cycles $a_z \in Z(R)$, almost all equal to 0, and for some $y \in L^u$. By construction, for each $z \in Z^{u+1}$ there is an $e_z \in L^{u+1}$ with $z = \partial(e_z)$, hence

$$x = \partial(\sum_{z\in Z^u} (-1)^{|a_z|} a_z e_z + y) \in L^{u+1} \subseteq L.$$

Thus, x is a boundary as well so $cls(x) = 0$ in $H(L)$ therefore ϵ is a quasi-isomorphism. By 2.9 the quasi-isomorphism ϵ is surjective if and only if it is surjective on cycles and by construction, this happens if and only if Z^0 generates the graded $Z(R)-$module $Z(M)$.

Constructive proof 3.37 ii. Before starting the main arguments some preparations are needed so we start by describing the concept of totaling.

Totaling Fix a complex of DG modules over R

$$\mathbf{X} = \cdots \longrightarrow X^{u-1} \xrightarrow{\delta^{u-1}} X^u \xrightarrow{\delta^u} X^{u+1} \longrightarrow \cdots$$

Lemma 3.39. *Let X be the DG module $\prod_{u\in\mathbb{Z}} \Sigma^{-u}(X^u)$ where elements of X are the families $(x^u)_{u\in\mathbb{Z}} \in X$ with $x^u \in \Sigma^{-u}X^u$, and the formulas*

$$r((x^u)_{u\in\mathbb{Z}}) = ((-1)^{|r|u} r x^u)_{u\in\mathbb{Z}}$$
$$\partial^X((x^u)_{u\in\mathbb{Z}}) = ((-1)^{|u|}\partial^{X^u}(x^u))_{u\in\mathbb{Z}}$$

describe its module structure and differential. Then the formula

$$\delta((x^u)_{u\in\mathbb{Z}}) = (\delta^{u-1}(x^{u-1}))_{u\in\mathbb{Z}}$$

defines an $R-$linear chain map $\delta : X \longrightarrow X$ of degree -1 such that $\delta^2 = 0$.

We define the totaling of the complex $\mathbf{X}$ to be the DG module

$$Tot(\mathbf{X}) = (X^\natural, \partial^X + \delta)$$

Construction For $u \leq -1$ define a morphism $\delta^{u-1} : I^{u-1} \longrightarrow I^u$ by setting $I^u = 0$ for all $u \leq -2$ and $I^{-1} = M$. If v is a non-negative integer, then assume by induction that morphisms δ^{u-1} have been chosen for $u < v$, and set $M^v = Coker(\delta^{v-2})$. Choose a surjective quasi-isomorphism $\epsilon^v : L^v \longrightarrow (M^v)^\vee$ of DG modules over R^o with L^v semi-free. Set $I^v = (L^v)^\vee$, define δ^{v-1} to be the composition

$$I^{v-1} \xrightarrow{\pi^{v-1}} M^v \xrightarrow{\zeta} (M^v)^{\vee\vee} \xrightarrow{(\epsilon^v)^\vee} (L^v)^\vee = I^v$$

where ζ is the natural evaluation and π^{v-1} is the canonical projection. Now form the sequences of DG modules

$$\mathbf{I} = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow I^0 \xrightarrow{\delta^0} \cdots \longrightarrow I^u \xrightarrow{\delta^u} I^{u+1} \longrightarrow \cdots$$

Let $I = Tot(\mathbf{I})$ be the totaling and define a map

$$\eta : M \mapsto I \text{ by } \eta(m) = (x^u)_{u\in\mathbb{Z}} \qquad \text{where } x^u = \begin{cases} \delta^{-1}(m) & \text{for } u = 0; \\ 0 & \text{for } u \neq 0. \end{cases}$$

It is clear from the formula that η is a morphism in $\mathcal{DGM}(R)$. One can show that the DG module I is homotopically injective and the graded $R^\natural-$module $I^\natural$ is injective and therefore I is semi-injective. By putting more effort one may see that η is an injective quasi-isomorphism.

Comparing semi-projective resolution and ordinary projective resolution Let A be a commutative ring and M be an $A-$module. A can be seen as a DGA concentrated in degree zero and also M can be seen as a DG module concentrated in degree zero.

Suppose

$$F : \cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow F_0 \xrightarrow{d_0} M \longrightarrow 0$$

is a projective resolution of M. Define the DG module (P, δ) such that $P_i = F_i$ where $i \geq 0$ and $F_i = 0$ for $i < 0$ and $\delta_i = d_i$ where $i > 0$ and $\delta_i = 0$ for $i \leq 0$. Now, define the map $\pi : P \longrightarrow M$ such that $\pi_0 = d_0$ and $\pi_i = 0$ for $i \neq 0$. It can be seen that π is a map of DG modules and it is a quasi-isomorphism and P is a semi-projective resolution of M as a DG module.

Remark 3.40. The above discussion lets us talk about semi-projective resolution and projective resolution interchangeably.

3.7 Cotorsion Theory and Model Categories

In this section, the notation of cotorsion theory is introduced based on the definitions and results of [13]. In addition we give an alternative proof for 3.22 by using cotorsion theory and its related results for model categories from [21].

Definition 3.41. In an abelian category $\mathcal{A}$, for a given class of object $\mathcal{C}$ let ${}^{\perp}\mathcal{C}$ denote the class of objects F such that $Ext_{\mathcal{A}}(F, C)=0$ for all $C \in \mathcal{C}$ and let $\mathcal{C}^{\perp}$ denote the class of objects G such that $Ext_{\mathcal{A}}(C, G)$=0 for all $C \in \mathcal{C}$. ${}^{\perp}\mathcal{C}$ and $\mathcal{C}^{\perp}$ are called orthogonal classes of $\mathcal{C}$.

Definition 3.42. A pair $(\mathcal{F}, \mathcal{C})$ of objects in $\mathcal{A}$ is called a cotorsion theory if $\mathcal{F}^{\perp}$=$\mathcal{C}$ and ${}^{\perp}\mathcal{C}$=$\mathcal{F}$. A class D is said to generate the cotorsion theory if ${}^{\perp}\mathcal{D}$=$\mathcal{F}$ (and so $\mathcal{D} \subset \mathcal{C}$) and a class $\mathcal{G}$ is said to cogenerate the cotorsion theory if $\mathcal{G}^{\perp}$=$\mathcal{C}$ (and so $\mathcal{G} \subset \mathcal{F}$).

Example 3.43. In the category of modules over a ring K, $(\mathcal{M}, \mathcal{I})$ and $(\mathcal{P}, \mathcal{M})$ are cotorsion theories where $\mathcal{M}$ denotes the class of all modules, $\mathcal{I}$ and $\mathcal{P}$ denote the class of injective and projective modules respectively.

Definition 3.44. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to have enough injectives if for every object X there is an exact sequence $0 \longrightarrow X \xrightarrow{i} C \xrightarrow{q} F \longrightarrow 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. Also we say it has enough projectives if for every X there is an exact sequence $0 \longrightarrow C \xrightarrow{i} F \xrightarrow{q} X \longrightarrow 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. Moreover, if a cotorsion theory has enough injective and projective objects we call it a complete cotorsion theory.

Remark 3.45. If $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory, then $\mathcal{F}$ and $\mathcal{C}$ are both closed under extensions and summands and $\mathcal{F}$ contains all projective objects while $\mathcal{C}$ contains all injective objects. Also $\mathcal{F}$ is closed under arbitrary coproduct if so is $\mathcal{A}$ and $\mathcal{C}$ is closed under arbitrary product if so is $\mathcal{A}$.

Definition 3.46. A nonempty subcategory of an abelian category is called thick if it is closed under retract, and whenever two out of three entries in a short exact sequence are in the thick subcategory, so is the third.

Definition 3.47. An abelian model category is a complete and cocomplete abelian category $\mathcal{A}$ equipped with a model structure such that

i. A map is a cofibration if and only if it is a monomorphism with cofibrant cokernel.

ii. A map is a fibration if and only if it is an epimorphism with fibrant kernel.

The next theorem is the main result of [21].

Theorem 3.48. *Suppose $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ are three classes of objects in a complete and cocomplete abelian category $\mathcal{A}$, such that*

- *$\mathcal{W}$ is thick.*
- *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion theories.*

Then there exists a unique abelian model structure on $\mathcal{A}$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of acyclic objects.

In the rest of this section, an alternative proof for 3.22 is given. We start with the next lemma which plays a central role. To be consistent, note that while using Theorem 3.37 regarding the semi-injective resolutions, the constructive proof must be taken into account otherwise the strange loop

$$\begin{tikzcd} & \text{Theorem 3.22} \arrow[dr, Rightarrow, bend left] & \\ \text{Theorem 3.50} \arrow[ur, Rightarrow, bend left] & & \text{Theorem 3.37} \arrow[ll, Rightarrow, bend left] \end{tikzcd}$$

will happen.

Lemma 3.49. *In $\mathcal{DGM}(R)$, the pair $(\mathcal{W}, \mathcal{F})$ is a complete cotorsion theory where $\mathcal{W}$ denotes the class of all quasi-trivial objects and $\mathcal{F}$ is the class of all semi-injective objects.*

Proof. The pair $(\mathcal{W}, \mathcal{F})$ is a cotorsion theory if and only if (a) $I \in \mathcal{F} \Leftrightarrow Ext_{\mathcal{A}}(w, I)$=0 for all $w \in \mathcal{W}$ and (b) $w \in \mathcal{W} \Leftrightarrow Ext_{\mathcal{A}}(w, I)$=0 for all $I \in \mathcal{F}$.

b $\Rightarrow$) Let $w \in \mathcal{W}$, for a given $I \in \mathcal{F}$ consider the arbitrary short exact sequence $0 \longrightarrow I \xrightarrow{i} X \xrightarrow{\pi} w \longrightarrow 0$ where $X \in \mathcal{DGM}(R)$. As $H(w) = 0$ then i must be an injective quasi-isomorphism hence by 2.75 it has a left inverse which means $Ext_{\mathcal{A}}(w, I)$=0

b $\Leftarrow$) Suppose $w \in \mathcal{DGM}(R)$ and $Ext_{\mathcal{A}}(w, I)$=0 for all $I \in \mathcal{F}$. By Theorem 3.37 we can assume that $\beta : w \longrightarrow \breve{w}$ is a semi-injective resolution for w where $\breve{w} \in \mathcal{F}$. Since $Ext_{\mathcal{A}}(w, \Sigma^{-1}\breve{w}) = 0$, the short exact sequence of the mapping cone of β will split and hence $Cone\Sigma^{-1}\beta \cong \Sigma^{-1}\breve{w} \oplus w$. In addition $Cone\Sigma^{-1}\beta$ is a quasi-trivial object because β is a quasi-isomorphism which shows w is a quasi-trivial object and belongs to $\mathcal{W}$.

a $\Rightarrow$) Let $I \in \mathcal{F}$ for a given $w \in \mathcal{W}$. Follow the argument for the first part of (b).

a $\Leftarrow$) Let $I \in \mathcal{DGM}(R)$, for a given injective quasi-isomorphism $i : I \longrightarrow M$ we can see that $M/I \in \mathcal{W}$ so by assumption $Ext_{\mathcal{A}}(M/I, I)$=0 which means i has a left inverse and therefore $I \in \mathcal{F}$ by 2.75.

For a given object X, let $\beta : X \overset{\simeq}{\hookrightarrow} \check{X}$ be its semi-injective resolution. Then the short exact sequence $0 \longrightarrow X \xrightarrow{\beta} \check{X} \longrightarrow \check{X}/X \longrightarrow 0$ shows that the theory has enough injectives while $0 \longrightarrow \Sigma^{-1}\check{X} \longrightarrow Cone\Sigma^{-1}\beta \longrightarrow X \longrightarrow 0$ shows enough projectives exist. □

Theorem 3.50. *In $\mathcal{DGM}(R)$, suppose $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ are classes of all objects, semi-injective objects and quasi-trivial objects, respectively. Then there exists a unique abelian model structure on $\mathcal{DGM}(R)$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of acyclic objects.*

Proof. It is clear that $\mathcal{W}$ is thick and $\mathcal{F} \cap \mathcal{W}$ is the class of all injective objects. Therefore $\mathcal{C}$ is the class of all objects so $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a complete cotorsion theory. Additionally, $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a complete cotorsion theory by Lemma 3.49. Hence by 3.48 we have a unique abelian model structure on $\mathcal{DGM}(R)$.

□

4 Functors between $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$

In chapter 3, we have defined two model structures on the category of differential graded modules so one can talk about the derived category of differential graded modules. One way of understanding a new derived category is to compare it with a known one. Firstly in this chapter, we introduce the functors extension, restriction and co-extension of scalars between two categories of differential graded modules. It is worth noting that these functors are widely used in commutative algebra and stable equivariant homotopy theory while analyzing algebraic models of change of groups (a usage of these functors can be found in [16, Sect 9]). Then abstract properties of these functors, considered as functors between two model categories, are provided and an application in the case of change of ring maps is given.

4.1 Introducing Functors between $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$

For a given morphism of DGAs $\Theta : R \longrightarrow S$ the following series of functors can be defined.

$$\begin{tikzcd}
\mathcal{DGM}(R) \arrow[r, shift left, "\Theta_*"] \arrow[r, bend right=30, "\Theta_!"] \arrow[r, bend right=50, "\Theta_!^{\delta}"] & \mathcal{DGM}(S) \arrow[l, shift left, "\Theta^*"] \arrow[l, bend left=70, "\Theta^!_{\delta}"]
\end{tikzcd}$$

$$\Theta^*(N_S)=N_R$$
$$\Theta_*(M_R)={}_SS_R \otimes_R M_R$$
$$\Theta_!(M_R)=Hom_R({}_SS_R, M_R)$$
$$\Theta_!^{\delta}(M_R) = {}_S\delta_R \otimes_R M_R$$
$$\Theta^!_{\delta}(N_S) = Hom_S({}_S\delta_R, N_S)$$

where δ is an R–S DG-bimodule. Note that the map Θ makes the ring S into a $R-$module and as a result $Hom_R(S, R)$ becomes an R–S DG-bimodule and will be used to determine more properties of δ. The above diagram and equations are our conventions for the rest of this chapter and we also assume that R and S are commutative DGAs over K.

The next proposition is a generalization of standard theorems regarding extension, restriction and co-extension of scalars for which proofs can be found in [8]. Although the proof of next proposition is standard, we include the proof since we use some details especially the diagrams later to verify Quillen equivalence criteria.

Proposition 4.1. *The following pairs of functors are adjoint.*

i. Θ_* *is left adjoint to* Θ^**.*

ii. Θ^* *is left adjoint to* $\Theta_!$*.*

iii. $\Theta_!^{\delta}$ *is left adjoint to* $\Theta^!_{\delta}$*.*

Proof. Suppose M is an arbitrary $R-$module and N is an arbitrary $S-$module.

(i) To prove the first statement it is needed to show that

$$Hom_S(S \otimes_R M, N) \cong Hom_R(M, N)$$

which is equal to say that for a given $R-$module map $f : M \longrightarrow N$ there is a unique $S-$module map $g : S \otimes_S M \longrightarrow N$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"] & R \otimes_R M \arrow[r, "\Theta\otimes 1_M"] & S \otimes_R M \arrow[dll, dotted, "g"] \\ N & & \end{tikzcd} \tag{4.1.1}$$

If the map g exists it must satisfy

$$g(s \otimes m) = g(s._R 1_R \otimes m) = g(s._S 1_S \otimes m) = sg(\Theta(1_R) \otimes m) = sf(m)$$

where 1_R and 1_S denote the unities in R and S, and $._R$ and $._S$ denote the multiplications in R and S. The above relation shows how to define g and why it is unique. Θ_* is called extension of scalars and Θ^* is called restriction of scalars functor. In addition, if $M = N$ and $f = 1_N$ then there is a canonical $S-$module map, $g : S \otimes_R N \longrightarrow N$ such that $g(s \otimes n) = sn$. The diagram 4.1.1 shows this map is a surjection and it has a right inverse as a map of $R-$modules.

(ii) To prove the second statement it is needed to show that

$$Hom_S(N, Hom_R(S, M)) \cong Hom_R(N, M)$$

which is equivalent to saying that for a given $R-$module map $f : N \longrightarrow M$ there is a unique $S-$module map $g : N \longrightarrow Hom_R(S, M)$ such that the following diagram commutes.

$$\begin{tikzcd} & Hom_R(S, M) \arrow[d, "Hom_R(\Theta, M)"] \\ & Hom_R(R, M) \arrow[d, "ev(1)"] \\ N \arrow[uur, dotted, "g"] \arrow[r, "f"'] & M \end{tikzcd} \tag{4.1.2}$$

First of all the DG module $Hom_R(S, M)$ has the structure of $S-$module by $(s'f)(s) = f(s's)$ for $f \in Hom_R(S, M)$ and $s', s \in S$. For simplicity define

$$\hat{\Theta} := ev \circ Hom_R(\Theta, M)$$

therefore for any $h \in Hom_R(S, M)$ $\hat{\Theta}(h) = h(\Theta(1_R))$. If g exists it must satisfy $sg(n) = g(sn)$ as it is a $S-$module map and the following relation must hold.

$$g(n)(s) = sg(n)(\Theta(1)) = g(sn)(\Theta(1)) = \hat{\Theta}(g(sn)) = f(sn).$$

The above relation shows how to define g and why it is unique. $\Theta_!$ is called co-extension of scalars. In addition, if $M = N$ and $f = 1_N$ then there is a canonical $S-$module map, $g : N \longrightarrow Hom_R(S, N)$ such that $g(n)(s) = sn$. The diagram 4.1.2 shows this map is an injection and as $R-$module map it has a left inverse.

(iii) To prove the last statement it is needed to show that

$$Hom_S(M \otimes_R \delta, N) \cong Hom_R(M, Hom_S(\delta, N))$$

which has been proved in 2.24. □

4.2 Quillen Pairs and Quillen Equivalence between $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$

In this section, we introduce necessary and sufficient conditions under which an adjunction turns out to be a Quillen pair or Quillen equivalence. However some preparations are essential during our main arguments.

Proposition 4.2. *For each DG module M over R the following hold.*

i. For a quasi-isomorphism, $\alpha : P \longrightarrow P'$, of homotopically projective DG modules, the maps $Hom_R(M, \alpha)$ and $Hom_R(\alpha, M)$ are homotopy equivalences.

ii. For a quasi-isomorphism, $\beta : I \longrightarrow I'$, of homotopically injective DG modules, the maps $Hom_R(M, \beta)$ and $Hom_R(\beta, M)$ are homotopy equivalences.

iii. For a quasi-isomorphism, $\gamma : F \longrightarrow F'$, of homotopically flat DG modules, the map $\gamma \otimes_R M$ is a quasi-isomorphism.

Proof. (i) First suppose P and P' are semi-projective and consider the projective model on $\mathcal{DGM}(R)$. As semi-projective modules are fibrant-cofibrant objects α is a homotopy equivalence by 1.9. If P and P' are homotopically projective then they are homotopic to semi-projective modules $\hat{P}$ and $\hat{P'}$ and therefore we can build the commutative diagram

$$\begin{tikzcd} \hat{P} \arrow[r, "\hat{\alpha}"] \arrow[d, "\pi_p"'] & \hat{P'} \arrow[d, "\pi_{p'}"] \\ P \arrow[r, "\alpha"'] & P' \end{tikzcd}$$

where $\hat{\alpha}$ exists because $\pi_{P'}$ is a surjective quasi-isomorphism and $\hat{P'}$ is semi-projective. Considering the fact that $\hat{\alpha}$ is a quasi-isomorphism between semi-projective modules we can conclude that it is a homotopy equivalence. Hence, α is a homotopy equivalence because all the rest of three maps in the commutative diagram are homotopy equivalences. Thus, the maps $Hom_R(M, \alpha)$ and $Hom_R(\alpha, M)$ are homotopy equivalences.

(ii) A dual argument for (i) leads to the desired result.

(iii) By (ii) the map $\gamma^\vee : F'^\vee \longrightarrow F^\vee$ is a homotopy equivalence of homotopically injective modules. Hence, $Hom_R(M, \gamma^\vee)$ is a quasi-isomorphism. Because we have an isomorphism of functors between $Hom_R(M, \gamma^\vee)$ and $(\gamma \otimes_R M)^\vee$, $(\gamma \otimes_R M)^\vee$ is a quasi-isomorphism and therefore $(\gamma \otimes_R M)$ is a quasi-isomorphism. □

Quillen Pair

Theorem 4.3. *Let ${}_S\mathcal{A}_R$ be a R–S DG-bimodule and consider the projective model on $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$, then the left adjoint*

$$ {}_S\mathcal{A}_R \otimes - : \mathcal{DGM}(R) \longrightarrow \mathcal{DGM}(S)$$

is a left Quillen functor if and only if ${}_S\mathcal{A}_R$ is a semi-projective $S-$module.

Proof. Let ${}_S\mathcal{A}_R \otimes -$ be a left Quillen functor, then it preserves all (trivial)cofibrations. Considering $i : R \longrightarrow Cone(1_R)$ as a cofibration in $\mathcal{DGM}(R)$ its image is a cofibration in $\mathcal{DGM}(S)$ as well. Hence the natural map ${}_S\mathcal{A}_R \longrightarrow Cone(1_{{}_S\mathcal{A}_R})$ is an injection with semi-projective cokernel and therefore $\Sigma\, {}_S\mathcal{A}_R$ and ${}_S\mathcal{A}_R$ are semi-projective $S-$modules.

Let ${}_S\mathcal{A}_R$ be a DG $R-$module which is also a semi-projective $S-$module. Since $\mathcal{DGM}(R)$ is a cofibrantly generated model category we just need to show that the image of elements $I = \{ S^{n-1} \longrightarrow D^n \}$ and $J = \{ 0 \longrightarrow D^n \}$ for all $n \in \mathbb{Z}$ are cofibrations and trivial cofibrations respectively. For a given $i_n \in I$ there is a linearly split short exact sequence of DG $R-$module, $0 \longrightarrow S^{n-1} \xrightarrow{i_n} D^n \longrightarrow S^n \longrightarrow 0$ so applying the functor ${}_S\mathcal{A}_R \otimes -$ yields a linearly split short exact sequence of DG $S-$modules and its isomorphic sequence are as below by 2.40.

$$\begin{tikzcd}
0 \arrow[r] & {}_S\mathcal{A}_R \otimes S^{n-1} \arrow[r, "1\otimes i"] \arrow[d, Rightarrow] & {}_S\mathcal{A}_R \otimes D^n \arrow[r] \arrow[d, Rightarrow] & {}_S\mathcal{A}_R \otimes S^n \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & \Sigma^n {}_S\mathcal{A}_R \arrow[r, "\overline{1\otimes i}"] & Cone(1_{\Sigma^n {}_S\mathcal{A}_R}) \arrow[r] & \Sigma^{n+1} {}_S\mathcal{A}_R \arrow[r] & 0.
\end{tikzcd}$$

Due to the exactness of the second row, $\overline{1 \otimes i}$ is an injective map with $\Sigma^{n+1}{}_S\mathcal{A}_R$ as its cokernel, which is a semi-projective DG module by assumption. Hence $1 \otimes i$ is a cofibration (an injection with semi-projective cokernel). For a given map in J, since ${}_S\mathcal{A}_R \otimes D^n \cong Cone(1_{\Sigma^n {}_S\mathcal{A}_R})$ is contractible, $0 \longrightarrow Cone(1_{\Sigma^n {}_S\mathcal{A}_R})$ is a trivial cofibration. $\square$

Corollary 4.4. *Consider the projective model structure on $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$; the pair (Θ_*, Θ^*) is a Quillen pair. In addition, the pair $(\Theta_!^\delta, \Theta_\delta^!)$ is a Quillen pair if and only if δ is a semi-projective $S-$module.*

Remark 4.5. By the classical definition, the derived functor of $Hom_R(A, -)$ is equal to $Hom_R(\hat{A}, -)$ where $\hat{A}$ is the projective resolution of A. By considering the projective model structure and the new definition of derived functor talking about the derived functor of the functor $Hom_R(A, -)$ is meaningful if A already is a cofibrant object so the classical definition and the new definition are compatible.

Proposition 4.6. *Let ${}_S\mathcal{A}_R$ be a R–S DG-bimodule, then for the left adjoint*

$${}_S\mathcal{A}_R \otimes - : \mathcal{DGM}(R) \longrightarrow \mathcal{DGM}(S)$$

between the categories of DG $R-$modules and DG $S-$modules with the injective model structure on both, the functor ${}_S\mathcal{A}_R \otimes -$ is a left Quillen functor if and only if ${}_S\mathcal{A}_R$ is a semi-flat $R-$module.

Proof. The functor $\mathcal{A} \otimes_R -$ is a left Quillen functor if and only if it sends all injective maps to injective maps and preserves all injective quasi-isomorphism which means that ${}_S\mathcal{A}_R$ is a semi-flat DG $R-$module. $\square$

Corollary 4.7. *Consider the injective model structure on both $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$, then the pair $(\Theta^*, \Theta_!)$ is a Quillen pair. In addition, the pair $(\Theta_!^\delta, \Theta_\delta^!)$ is a Quillen pair if and only if δ is a semi-flat $R-$module.*

Quillen Equivalence In the rest of this section, we analyze the conditions under which a Quillen pair turns out to be a Quillen equivalence.

Proposition 4.8. *Consider the projective model structure on both $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$; the Quillen adjunction $(\Theta_*, \Theta^*, \phi)$ is a Quillen equivalence if and only if Θ is a quasi-isomorphism.*

Proof. Suppose $\Theta : R \longrightarrow S$ is a quasi-isomorphism. For a given morphism $f \in Hom_R(\Theta_*(M), N)$ where $M \in \mathcal{DGM}(R)$ and $N \in \mathcal{DGM}(S)$ the diagram

$$\begin{tikzcd} S \otimes M \arrow[r, "f"] & N \arrow[d, "1_N"] \\ R \otimes M \arrow[u, "\Theta\otimes 1_M"] \arrow[r, "\phi(f)"'] & N \end{tikzcd}$$

commutes. If M is a semi-projective DG module (cofibrant object), by 2.83 and 4.2 the map $\Theta \otimes 1_M$ is a quasi-isomorphism. Thus, the map f is a quasi-isomorphism if and only if $\phi(f)$ is a quasi-isomorphism.

Suppose $(\Theta_*, \Theta^*, \phi)$ is a Quillen equivalence. For the cofibrant object R the map $S \otimes_R R \xrightarrow{1} S$ is a quasi-isomorphism and therefore the map, $\Theta = \phi(1) : R \longrightarrow S$, is a quasi-isomorphism as well.

$\square$

Proposition 4.9. *Consider the injective model structure on both $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$; the Quillen adjunction $(\Theta^*, \Theta_!, \phi)$ is a Quillen equivalence if and only if Θ is a quasi-isomorphism.*

Proof. Suppose $\Theta : R \longrightarrow S$ is a quasi-isomorphism. For a given morphism $f \in Hom_R(\Theta^*(N), M)$ where $M \in \mathcal{DGM}(R)$ and $N \in \mathcal{DGM}(S)$ the following diagram commutes.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "1_N"'] & Hom_R(R,M) \\ N \arrow[r, "\phi(f)"'] & Hom_R(S,M) \arrow[u, "Hom_R(\Theta,M)"'] \end{tikzcd} \tag{4.2.1}$$

If M is a semi-injective DG module (fibrant object), by 2.74 and 2.75 the map $Hom_R(\Theta, M)$ is a quasi-isomorphism. Hence the map f is a quasi-isomorphism if and only if $\phi(f)$ is a quasi-isomorphism.

Suppose $(\Theta^*, \Theta_!, \phi)$ is a Quillen equivalence and the maps $i : R \longrightarrow \breve{R}$ and $j : S \longrightarrow \breve{S}$ are semi-injective resolutions (fibrant replacement) of R and S in the $\mathcal{DGM}(R)$. By assumption, the diagram 4.2.1 shows that the following homomorphisms are quasi-isomorphisms. In fact, the right vertical map is a quasi-isomorphism for every fibrant object I by Lemma 1.27.

$$Hom_R(S, \breve{R}) \xrightarrow{Hom_R(\Theta, \breve{R})} Hom_R(R, \breve{R}) \tag{4.2.2}$$

$$Hom_R(S, \breve{S}) \xrightarrow{Hom_R(\Theta, \breve{S})} Hom_R(R, \breve{S}) \tag{4.2.3}$$

Furthermore, as i is an injective quasi-isomorphism and $\breve{S}$ is semi-injective the map $\breve{\Theta}$ exists and the square in the diagram below commutes

$$\begin{tikzcd} R \arrow[r, "\Theta"] \arrow[d, "i"'] & S \arrow[d, "j"] \arrow[dl, dotted, "h"] \\ \breve{R} \arrow[r, "\breve{\Theta}"'] & \breve{S} \end{tikzcd} \tag{4.2.4}$$

The quasi-isomorphism 4.2.2 shows that there exists a unique morphism h up to homotopy such that $h\Theta \sim i$ showing $H(\Theta)$ is an injective map. The upper triangle commutes up to homotopy so $\breve{\Theta}i \sim \breve{\Theta}h\Theta \sim j\Theta$, additionally the quasi-isomorphism 4.2.3 shows that $\breve{\Theta}h \sim j$. Hence $H(\breve{\Theta})$ is a surjection which means $H(\Theta)$ is a surjection as well.

□

For the two next propositions, an extra condition is imposed. In fact we assume that S is a semi-projective $R-$module. Note that this extra condition has no effect on results at the derived level as long as S has a semi-free resolution with an $R-$algebra structure [12].

Proposition 4.10. *Suppose S is a semi-projective $R-$module and $\delta = Hom_R(S, R)$, with the injective model structure on both $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$, the Quillen adjunction $(\Theta_!^\delta, \Theta_\delta^!, \phi)$ is a Quillen equivalence if Θ is a quasi-isomorphism.*

Proof. By assumption Θ is a quasi-isomorphism of semi-projective modules and therefore it is a homotopy equivalence by 4.2. Assume $Hom_R(\Theta, R) = \bar{\Theta} : \delta \longrightarrow R$ is the induced quasi-isomorphism. First of all, as $(\Theta_!^\delta, \Theta_\delta^!, \phi)$ is a Quillen adjunction, the DG module δ must be a semi-flat $R-$module and therefore $\bar{\Theta}$ is a quasi-isomorphism of semi-flat $R-$modules. Hence, by 4.2 for every $R-$module M the following map is a quasi-isomorphism.

$$1_M \otimes_R \bar{\Theta} : M \otimes_R \delta \longrightarrow M \otimes_R R \tag{4.2.5}$$

Furthermore, for every semi-injective $S-$module N and the composition quasi-isomorphism $\Theta \circ \bar{\Theta} = \underline{\Theta} : \delta \longrightarrow S$, the following map is a quasi-isomorphism by 2.74.

$$Hom_S(\underline{\Theta}, N) : Hom_S(S, N) \longrightarrow Hom_S(\delta, N) \tag{4.2.6}$$

The diagram below, 4.2.5 and 4.2.6 show that for a semi-injective $S-$module (fibrant) N and an arbitrary $R-$module M, the map β is a quasi-isomorphism if and only if $\phi(\beta)$ is a quasi-isomorphism.

$$\begin{tikzcd}
M \otimes_R \delta \arrow[r, "\beta"] \arrow[d, "1_M \otimes_R \bar{\Theta}"'] & Hom_S(S, N) \arrow[d, "Hom_S(\underline{\Theta}, N)"] \\
M \otimes_R R \arrow[r, "\phi(\beta)"] & Hom_S(\delta, N)
\end{tikzcd} \tag{4.2.7}$$

□

Proposition 4.11. *Suppose S is a semi-projective $R-$module and $\delta = Hom_R(S, R)$, with the projective model structure on both $\mathcal{DGM}(R)$ and $\mathcal{DGM}(S)$, the Quillen adjunction $(\Theta_!^\delta, \Theta_\delta^!, \phi)$ is a Quillen equivalence if Θ is a quasi-isomorphism.*

Proof. Suppose $\underline{\Theta} : \delta \longrightarrow S$ is the introduced quasi-isomorphism in 4.10. Since $(\Theta_!^\delta, \Theta_\delta^!, \phi)$ is a Quillen adjunction, the DG module δ must be a semi-projective $S-$module. Therefore by 4.2 for every $S-$module N the following map is a quasi-isomorphism.

$$Hom_S(\underline{\Theta}, N) : Hom_S(S, N) \longrightarrow Hom_S(\delta, N) \tag{4.2.8}$$

Furthermore, for every semi-projective $R-$module M and the induced quasi-isomorphism $\bar{\Theta} : \delta \longrightarrow R$, the following map is a quasi-isomorphism by 2.83 and 2.84.

$$1_M \otimes_R \bar{\Theta} : M \otimes_R \delta \longrightarrow M \otimes_R R. \tag{4.2.9}$$

The diagram 4.2.7 and the quasi-isomorphisms 4.2.8 and 4.2.9 show that for a semi-projective $R-$module (cofibrant) M and an arbitrary $S-$module N, the map β is a quasi-isomorphism if and only if $\phi(\beta)$ is a quasi-isomorphism. □

4.3 Relation between $\Theta_!$ and $\Theta_!^\delta$

The next Lemma is a special case of [2, Theorem 2]. Note that, although some results from [2] have been used to define the morphism, the employed techniques are different. Additionally, the finiteness conditions, introduced in Section 2.9, are used. Furthermore, this Lemma can be proved by using the results about the triangulated categories.

Lemma 4.12. *For a given small semi-free DG module, L and an arbitrary DG module M in $\mathcal{DGM}(R)$, the morphism.*

$$\mu_L : Hom_R(L, R) \bigotimes\nolimits_R M \longrightarrow Hom_R(L, M)$$
$$f \otimes m \longrightarrow \phi_{f,m} : (l \to (-1)^{|m||l|} f(l).m)$$

is a natural isomorphism. Furthermore, for any retract P of L, the map μ_P is an isomorphism. Conversely, if for a given P the map, μ_P is an isomorphism for every M then P is linearly projective and P is finitely presented.

Proof. The proof consists of three steps. First of all, if L is a finite direct sum of suspensions of R, $L = \bigoplus_{i=1}^{n} \Sigma^{n_i} R$, it is clear that μ_L is an isomorphism.

Secondly, we claim that for a linearly split short exact sequence,

$$0 \longrightarrow L_0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow 0 \tag{4.3.1}$$

if μ_{L_0} and μ_{L_2} are isomorphisms then μ_{L_1} is an isomorphism as well. By applying functors $Hom_R(-,R)\bigotimes_R M$ and $Hom_R(-,M)$ on the 4.3.1 the following commutative diagram is achieved in which rows are exact because 4.3.1 is linearly split.

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(L_0,R)\bigotimes_R M \arrow[r] \arrow[d, "\mu_{L_0}"] & Hom_R(L_1,R)\bigotimes_R M \arrow[r] \arrow[d, "\mu_{L_1}"] & Hom_R(L_2,R)\bigotimes_R M \arrow[r] \arrow[d, "\mu_{L_2}"] & 0 \\
0 \arrow[r] & Hom_R(L_0,M) \arrow[r] & Hom_R(L_1,M) \arrow[r] & Hom_R(L_2,M) \arrow[r] & 0
\end{tikzcd}$$

By assumption μ_{L_0} and μ_{L_2} are isomorphisms and therefore the five lemma shows that μ_{L_1} is an isomorphism.

In the general case, suppose that $0 \subseteq L_0 \subseteq L_1 \subseteq ... \subseteq L_n$ is the finite filtration for L such that L_i/L_{i+1} is a finite direct sum of suspensions of R. Employing the two previous steps and induction leads to the desired result, as long as we know that

$$0 \longrightarrow L_i \longrightarrow L_{i+1} \longrightarrow L_i/L_{i+1} \longrightarrow 0$$

is a linearly split exact sequence.

For a given map $g : M \longrightarrow M'$, consider the diagram

$$\begin{tikzcd}
Hom_R(L,R)\bigotimes_R M \arrow[r, "\mu_L"] \arrow[d, "1\otimes g"'] & Hom_R(L,M) \arrow[d, "\tilde{g}"] \\
Hom_R(L,R)\bigotimes_R M' \arrow[r, "\mu_L"'] & Hom_R(L,M').
\end{tikzcd} \tag{4.3.2}$$

For an element of $Hom_R(L,R)\bigotimes_R M$, we have

$$f \otimes m \xrightarrow{\mu_L} \phi_{f,m} \xrightarrow{\tilde{g}} g \circ \phi_{f,m} : (t \rightarrow g(f(t).m))$$

also

$$f \otimes m \xrightarrow{1\otimes g} f \otimes g(m) \xrightarrow{\mu_L} \phi_{f,g(m)}$$

which shows that the diagram 4.3.2 is commutative due to the fact that $\phi_{f,g(m)} = g \circ \phi_{f,m}$. Therefore the map μ_L is a natural map. To show that for the retract P of L the map μ_P is an isomorphism, apply two functors $Hom_R(-,R)\bigotimes_R M$ and $Hom_R(-,M)$ on the retract diagram, then the naturality of μ_L and μ_P yield the commutative diagram

$$\begin{tikzcd}
Hom_R(P,R)\bigotimes_R M \arrow[rr, "\mu_P"] \arrow[dd, "1"'] \arrow[dr, "i_1"] & & Hom_R(P,M) \arrow[dd] \arrow[dr, "i_2"] & \\
& Hom_R(L,R)\bigotimes_R M \arrow[rr, "\mu_L"] \arrow[dl, "\pi_1"] & & Hom_R(L,M) \arrow[dl, "\pi_2"] \\
Hom_R(P,R)\bigotimes_R M \arrow[rr, "\mu_P"'] & & Hom_R(P,M) &
\end{tikzcd} \tag{4.3.3}$$

As μ_L is an isomorphism, we have $i_1 = {\mu_L}^{-1} i_2 \mu_P$ and $\pi_2 = \mu_P \pi_1 {\mu_L}^{-1}$ and therefore $\pi_1 {\mu_L}^{-1} i_2$ is a left and $\pi_1 {\mu_L}^{-1} i_2$ is a right inverse for μ_P.

To prove the converse part, suppose μ_P is an isomorphism then it shows that the functor $Hom_R(P,-)$ is an right exact functor because $Hom_R(P,R)\bigotimes_R -$ is an right exact functor. In addition, for a given directed family $\{M_i\}_{i\in I}$, the following sequence of isomorphisms exists

$$\begin{aligned}
Hom_R(P, \varinjlim M_i) &\cong Hom_R(P,R) \bigotimes_R \varinjlim M_i \\
&\cong \varinjlim Hom_R(P,R) \bigotimes_R M_i \\
&\cong \varinjlim Hom_R(P, M_i)
\end{aligned}$$

which means that P is finitely presented and therefore it is a finite DG module.

□

Corollary 4.13. *Suppose R is a non-negative DG algebra, for a given module P the map μ_P is an isomorphism if and only if P is a finite semi-projective DG $R-$module.*

Proof. Suppose μ_P is an isomorphism, then 4.12 shows that P is a finite linearly projective DG $R-$module and therefore P is bounded below by 2.89. Since $P^\natural$ is projective over $R^\natural$ and P is bounded below, Theorem 2.94 shows that it is a semi-projective module. □

Combining Lemma 4.12 and Corollary 4.13 yields the next theorem.

Theorem 4.14. *There is a natural transformation* $\mu^* : \Theta_!^\delta \longrightarrow \Theta_!$ *if and only if there is a* $R{-}S{-}$*bimodule map* $\mu : \delta \longrightarrow Hom_R(S, R)$ *. Furthermore if* μ *is an isomorphism and* S *is a small semi-projective* $R-$*module then* μ^* *is an isomorphism and the converse is true if* R *is a non-negative DG algebra.*

Corollary 4.15. *Suppose* S *and* δ *are small semi-projective* $R-$*modules. If* μ *is a quasi-isomorphism then* $\mu_M^* : \Theta_!^\delta(M) \longrightarrow \Theta_!(M)$ *is a homotopy equivalence.*

Proof. By assumption and Lemma 4.18 $Hom_R(S, R)$ is a semi-projective $R-$module and therefore $\mu : \delta \longrightarrow Hom_R(S, R)$ is a homotopy equivalence. Hence in the diagram

$$\delta \otimes M \xrightarrow{\mu\otimes M} Hom_R(S, R) \otimes M \xrightarrow{\mu_S} Hom_R(S, M)$$

the map $\mu \otimes M$ is a homotopy equivalence and μ_S is an isomorphism and therefore the composition is a homotopy equivalence. □

4.4 Relation between Θ^* and $\Theta_\delta^!$

In this section, an analysis of the relation between Θ^* and $\Theta_\delta^!$ is given. But before starting the main argument we bring [2, Theorem 1] here as 4.16 and some other lemmas.

Theorem 4.16. *Let* L *be a DG* $R-$*module,* M *a DG* R–$S-$*bimodule and* N *a DG* $S-$*module. Then there is a canonical morphism*

$$\omega : L \otimes_R Hom_S(M, N) \longrightarrow Hom_S(Hom_R(L, M), N) \tag{4.4.1}$$

given by $\omega(l \otimes \lambda)(\gamma) = (-1)^{|l|(|\lambda|+|\gamma|)}\lambda(\gamma(l))$.

It is an isomorphism if L *is a small semi-free DG module.*

Corollary 4.17. *In Theorem 4.16 the map* ω *is an isomorphism if* L *is a small semi-projective module.*

Proof. By forming a retract diagram like 4.3.3 for the map ω and a similar argument the result may be achieved. □

Lemma 4.18. *If P is a small semi-projective DG $R-$module then*

$$Hom_R(Hom_R(P,R),R) \cong P$$

and $Hom_R(P,R)$ is also a semi-projective module.

Proof. Substituting $L = P$, $M = N = R$ and $R = S$ in Theorem 4.16 leads to the desired isomorphism. Moreover, the sequence of isomorphisms, deriving from 4.16,

$$Hom_R(Hom_R(P,R),-) \cong P \otimes_R Hom_R(R,-) \cong P \otimes_R -$$

and the fact that every semi-projective module is semi-flat show that $Hom_R(P,R)$ is semi-projective. □

Theorem 4.19. *Suppose δ is a small semi-projective $S-$module, then there is a natural transformation $\mu^* : \Theta^!_\delta \longrightarrow \Sigma^{-n}\Theta^*$ if and only if there is a R–$S-$bimodule map $\mu : \Sigma^n S \longrightarrow \delta$. Furthermore μ is an isomorphism if and only if μ^* is an isomorphism.*

Note that the condition of semi-projectivity of δ is quite reasonable because of 4.4.

Proof. Suppose $\mu^* : \Theta^i_\delta \longrightarrow \Sigma^{-n}\Theta^*$ is a natural isomorphism between the two functors. By applying these two functors on the object S, the canonical map

$$Hom_S(\delta, S) \longrightarrow \Sigma^{-n} S$$

is obtained. Dualizing this map and Lemma 4.18 yields the map $\Sigma^n S \longrightarrow \delta$.

Conversely, by applying the functor $Hom_S(-,S)$ on the map $\Sigma^n S \longrightarrow \delta$ we obtain the map $Hom_S(\delta,S) \longrightarrow \Sigma^{-n}S$ and applying the functor $-\otimes_S N$ yields the morphism $Hom_S(\delta,S)\otimes_S N \longrightarrow \Sigma^{-n}N$. Combining the last morphism and the results from 4.12 yields the morphism $Hom_S(\delta,N) \longrightarrow \Sigma^{-n}N$. □

Remark 4.20. If μ is a quasi-isomorphism then

$$Hom_S(\delta,S) \xrightarrow{Hom_S(\mu,S)} \Sigma^{-n}S$$

is a homotopy equivalence. Therefore conducting a similar argument to the previous proof leads to the fact that $Hom_S(\delta,N) \xrightarrow{Hom_S(\mu,N)} \Sigma^{-n}N$ is a homotopy equivalence.

Definition 4.21. Let R and S be DGAs over K and $\Theta : R \longrightarrow S$ be a ring homomorphism, then Θ is a Gorenstein morphism if there exists $n \in \mathbb{Z}$ such that

$$\widehat{\Theta^*(S)} \sim Hom_R(\widehat{\Theta^*(S)}, \Sigma^n R)$$

where $\widehat{\Theta^*(S)}$ denotes a semi-projective resolution of $\Theta^*(S)$ as an $R-$module.

Remark 4.22. A number of results from [25] can be adopted by employing Definition 4.21.

In the following examples, the concept of semi-projective, semi-free and projective resolution are used interchangeably. See Remark 3.40.

Example 4.23. Suppose $K[x]$ is the polynomial ring over K and the degree of x, is a positive integer. The canonical projection map $\pi : K[x] \longrightarrow K$ makes K into a $K[x]$ module. To find a semi-free resolution $\hat{K}$ of K we employ the constructive method; described in 3.37. In this case $E^0 = \{e_1 || e_1| = 0\}$ and $L^0 \cong K[x]$. In the next step $E^1 = \{e_x || e_x| = |x| + 1\}$ and $L^1 = e_x K[x] \oplus K[x]$ where $d(e_x) = x$. A straightforward calculation shows that $H(\epsilon^1) = K$ and $Ker(H(\epsilon^1)) = 0$, and therefore $\hat{K} \cong \Sigma^{|x|+1} K[x] \oplus K[x]$ where $\oplus$ is the modules direct sum but not a sum as DG modules. We also have

$$Hom_{K[x]}(\hat{K}, K[x]) \cong K[x] \oplus \Sigma^{-|x|-1} K[x] \cong \widehat{\Sigma^{-|x|-1} K}$$

The next example shows that Definition 4.21 is not valid for all DG modules.

Example 4.24. Suppose $R = K[x, y]$ is the polynomial ring over K where $|x| = |y| = 1$ and $S = K[x, y]/(x^2, y^2, xy)$. The canonical projection $\pi : R \longrightarrow S$ makes S into a $K[x, y]$ module. In this case for the semi-free resolution of S we have

$$\hat{S} \sim R \oplus \Sigma^2 R^{\oplus 3} \oplus \Sigma^3 R^{\oplus 2}$$

Example 4.25. Suppose $R = K[x, y, z]$ is the polynomial ring over K where $|x| = |y| = |z| = 1$ and $S = K[x, y, z]/(yz, x)$. The canonical projection $\pi : R \longrightarrow S$ makes S into a $K[x, y, z]$ module. The free resolution of S is in form of

$$\hat{S} : \cdots \longrightarrow 0 \longrightarrow \Sigma^3 R \xrightarrow{d_2} \Sigma R \oplus \Sigma^2 R \xrightarrow{d_1} R \xrightarrow{d_0} 0 \longrightarrow \cdots$$

where the matrix representations of the differential maps are

$$d_1 = \begin{pmatrix} x & yz \end{pmatrix}$$
$$d_2 = \begin{pmatrix} -yz \\ x \end{pmatrix}$$

and therefore in the language of differential graded modules

$$\hat{S} \sim R \oplus \Sigma^3 R \oplus \Sigma^2 R \oplus \Sigma^5 R$$

and the differential map is

$$D = \begin{pmatrix} 0 & x & yz & 0 \\ 0 & 0 & 0 & -yz \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For the dual of $\hat{S}$ as a chain complex we have

$$Hom_R(\hat{S}, R) : \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{d_0} \Sigma^{-1} R \oplus \Sigma^{-2} R \xrightarrow{d_{-1}} \Sigma^{-3} R \xrightarrow{d_{-2}} 0 \longrightarrow \cdots$$

where the matrix representations of the differential maps are

$$d_{-1} = \begin{pmatrix} yz & -x \end{pmatrix}$$
$$d_0 = \begin{pmatrix} x \\ yz \end{pmatrix}$$

and as modules

$$Hom_R(\hat{S}, R) \sim \Sigma^{-5} R \oplus \Sigma^{-2} R \oplus \Sigma^{-3} R \oplus R$$

and one may check differentials to see that

$$Hom_R(\hat{S}, R) \sim \Sigma^{-5} \hat{S}$$

In Example 4.25, (yz, x) is a regular sequence and therefore its semi-free resolution is a Koszul complex. The next proposition is some sort of generalization of this example.

Proposition 4.26. *Fix a sequence* $\mathbf{x} = (x_1, \cdots, x_n)$*, where* R *is a commutative ring concentrated in degree zero. Let* $K(\mathbf{x})$ *be the Koszul complex of* $\mathbf{x}$ *and* $\Theta : R \longrightarrow K(\mathbf{x})$ *be the canonical embedding of DGAs, then* $Hom_R(K(\mathbf{x}), R) \cong \Sigma^n K(\mathbf{x})$*.*

Proof. If K_i=$K(\mathbf{x})_i$ then it is a free $R-$module with rank $\binom{n}{i}$ and the basis

$$\{e_{j_1} \wedge \cdots \wedge e_{j_i} | 1 \leq j_1 \leq \cdots \leq j_i \leq n\}$$

The map $K_{n-i} \otimes K_i \longrightarrow R$ given by $u \otimes v \mapsto u \wedge v$ describes a perfect pairing which induces an isomorphism

$$\sigma_i : K_{n-i} \longrightarrow Hom_R(K_i, R) = Hom_R(K(\mathbf{x}), R)_{-i}$$

which can be described as

$$e_{j_1} \wedge \cdots \wedge e_{j_{n-i}} \longrightarrow \pm e_{k_1} \wedge \cdots \wedge e_{k_i}$$

where $\{j_1, \cdots, j_{n-i}\} \sqcup \{k_1, \cdots, k_i\} = \{1, \cdots, n\}$ and the sign $(\pm)$ is the sign of permutation

$$\begin{pmatrix} 1 & \cdots & n-i & n-i+1 & \cdots & n \\ j_1 & \cdots & j_{n-i} & k_1 & \cdots & k_i \end{pmatrix}$$

Therefore the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_{n-i} \arrow[r] \arrow[d] & K_{n-i-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Hom_R(K_i, R) \arrow[r] & Hom_R(K_{i+1}, R) \arrow[r] & \cdots
\end{tikzcd}$$

commutes meaning that $\sigma = (\sigma_i)$ is a chain isomorphism.

□

Example 4.27. Suppose $R = K[x, y, z]$ is the polynomial ring over K where $|x| = |y| = |z| = 1$ and $S = K[x, y, z]/(x^2 - z^2, x^2 - y^2, xy, yz, xz)$. The canonical projection $\pi : R \longrightarrow S$ makes S into a $K[x, y, z]$ module. In this case for the semi-free resolution of S we have

$$\hat{S} \sim R \oplus \Sigma^2 R^{\oplus 5} \oplus \Sigma^3 R^{\oplus 5} \oplus \Sigma^5 R$$

and one may justify

$$Hom_R(\hat{S}, R) \sim \Sigma^{-5} R \oplus \Sigma^{-3} R^{\oplus 5} \Sigma^{-2} R^{\oplus 5} \oplus R \sim \Sigma^{-5} \hat{S}$$

The next example is some sort of generalization for 4.27 and the proof is similar to the proof of [22, 3.2]

Example 4.28. Suppose $\Theta : R \longrightarrow S$ is a map between commutative rings such that $Ext^i_R(S,R) = 0$ where $i \neq d$ and $Ext^d_R(S,R) \cong S$. We may then suppose the semi-free resolution of S has the form

$$0 \longrightarrow R \longrightarrow F_{d-1} \longrightarrow F_{d-2} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow S \longrightarrow 0 \tag{4.4.2}$$

Because F_js are free and the above sequence is exact and also $Ext^i_R(S,R) = 0$ where $i \neq d$ the sequence

$$0 \longrightarrow Hom_R(F_0,R) \longrightarrow \cdots \longrightarrow Hom_R(F_{d-1},R) \longrightarrow R \tag{4.4.3}$$

is exact and therefore it is a semi-free resolution of $Ext^d_R(S,R) \cong S$. Due to the fact that 4.4.2 and 4.4.3 are both semi-free resolutions of S they are homotopic and in fact $Hom_R(\hat{S},R) \sim \Sigma^d \hat{S}$.

The special case of this example happens when R is a regular local ring and S is a Gorenstein homomorphic image of R and Θ is the canonical projection [22, 3.2].

4.5 Dual of the Classifying Space of a Lie Subgroup

Analyzing the induced map between classifying spaces of the inclusion of a compact subgroup of a Lie group is the main subject of this section. Moreover, some results of this section provide a class of examples for Theorem 4.19 and Definition 4.21. During this section, all cohomology rings are over the rational field $\mathbb{Q}$.

Background Let H be a closed subgroup of G where G is a compact Lie group. One can show that the canonical projection $G \longrightarrow G/H$ is a fibration with fibre H (see [27, 4.3]). Therefore the sequence

$$H \longrightarrow G \longrightarrow G/H$$

is a fibration. Additionally, the following diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\iota"'] & H \arrow[r, "\iota"] \arrow[d] & G \arrow[d] \\
G \arrow[r] \arrow[d] & \mathbf{E}H \arrow[r] \arrow[d] & \mathbf{E}G \arrow[d] \\
G/H \arrow[r] & \mathbf{B}H \arrow[r, "\mathbf{B}\iota"] & \mathbf{B}G
\end{tikzcd}$$

in which $\mathbf{E}H$ and $\mathbf{E}G$ are total spaces of groups H and G and $\mathbf{B}H$ and $\mathbf{B}G$ are classifying spaces of groups H and G, commutes and one can deduce that

$$G/H \longrightarrow \mathbf{B}H \xrightarrow{\mathbf{B}\iota} \mathbf{B}G$$

is a fibration (see [27, 6.3]).

Example 4.29. Suppose $U(3)$ denotes the unitary group of 3×3 matrices and $T(3)$ denotes its maximal torus. Therefore we have the fibration

$$U(3)/T(3) \longrightarrow \mathbf{B}T(3) \xrightarrow{\mathbf{B}\iota} \mathbf{B}U(3)$$

in which $\mathbf{B}U(3)$ and $\mathbf{B}T(3)$ denote the related classifying spaces. It is known from [17, 3.22] and [10] that

$$H^*(\mathbf{B}T(3)) \cong \mathbb{Q}[t_1, t_2, t_3]$$

$$H^*(\mathbf{B}U(3)) \cong \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3]$$

and the ring morphism $\mathbf{B}\iota^* : H^*(\mathbf{B}U(3)) \longrightarrow H^*(\mathbf{B}T(3))$ maps the elements of $H^*(\mathbf{B}U(3))$ as follows

$$\sigma_1 \longmapsto t_1 + t_2 + t_3$$
$$\sigma_2 \longmapsto t_1t_2 + t_1t_3 + t_2t_3$$
$$\sigma_3 \longmapsto t_1t_2t_3.$$

For simplicity we denote $H^*(\mathbf{B}U(3))$ and $H^*(\mathbf{B}T(3))$ by R and S respectively. As an $R-$module S is a free module (regarding our terminology it is a module free on a basis of cycles). A bit of calculation shows that

$$S \cong_R R \oplus \Sigma^2 R^{\oplus 2} \oplus \Sigma^4 R^{\oplus 2} \oplus \Sigma^6 R$$

and therefore

$$Hom_R(S, R) \cong \Sigma^{-6} S$$

where $6 = dim(U(3)) - dim(T(3)) = 3^2 - 3$.

Example 4.30. Suppose T^m and T^n are m and n dimensional tori and $n > m$. Then by [17, 3.13]

$$H^*(\mathbf{B}T^n) \cong \mathbb{Q}[x_1, \cdots, x_n]$$

$$H^*(\mathbf{B}T^m) \cong \mathbb{Q}[x_1, \cdots, x_m].$$

If $\iota : T^m \longrightarrow T^n$ is an inclusion then the induced map of cohomology of classifying spaces $\mathbf{B}\iota^*$ is the projection and therefore

$$H^*(\mathbf{B}T^m) \cong \mathbb{Q}[x_1, \cdots, x_n]/\langle x_{i_1}, \cdots, x_{i_{n-m}}\rangle.$$

Considering the fact that $\{x_{i_1}, \cdots, x_{i_{n-m}}\}$ is a regular sequence, the projective resolution of $H^*(\mathbf{B}T^m)$ as $H^*(\mathbf{B}T^n)-$module is a Koszul complex by [11, 2.A2]. Hence

$$Hom_{H^*(\mathbf{B}T^n)}(\widehat{H^*(\mathbf{B}T^m)}, H^*(\mathbf{B}T^n)) \sim \Sigma^d \widehat{H^*(\mathbf{B}T^m)}$$

where $\widehat{H^*(\mathbf{B}T^m)}$ denotes the semi-projective resolution of $H^*(\mathbf{B}T^m)$ and

$$d = |x_{i_1}| + \cdots + |x_{i_{n-m}}| + n - m = -dim(T^n/T^m).$$

Note that $|x_j| = -2$.

Theorem 4.31. *Let G be a compact connected Lie group and H be a closed subgroup such that G/H is an orientable manifold. Then there exists $d \in \mathbb{Z}$ such that*

$$\mathbf{R}Hom_{H^*(\mathbf{B}G)}(H^*(\mathbf{B}H), H^*(\mathbf{B}G)) \cong_D \Sigma^d H^*(\mathbf{B}H)$$

where $\cong_D$ denotes the isomorphism in the derived category and coefficients are in $\mathbb{Q}$.

A proof based on equivariant homology is provided in [6, 6.8] at the chain complex level. However, as long as we know that if $H^*(X)$ is a polynomial then $H^*(X) \cong C^*(X)$ we can conclude the result. In the rest of this section, an algebraic proof for this theorem is given while extra conditions are imposed. Before starting the main discussion, some known results are restated here. A proof for the next theorem can be found in [27, Theorem 8.3] and [10, Sec. 5].

Theorem 4.32. *Let G be a compact connected Lie group, H be a closed subgroup and rankG=rankH. Then $H^*(\mathbf{B}H)$ is a free and finitely generated $H^*(\mathbf{B}G)-$module and*

$$H^*(\mathbf{B}H) \cong H^*(\mathbf{B}G) \otimes_{\mathbb{Q}} H^*(G/H).$$

Theorem 4.33. *[29, Sec. 13 Theorem 2] If G is a connected compact Lie group and H is a connected compact subgroup, then the Euler characteristic $\chi(G/H) \geq 0$. Moreover $\chi(G/H) > 0$ if and only if the rank of G equals the rank of H.*

Lemma 4.34. *Suppose R, S and T are DGAs and maps $R \xrightarrow{f} S \xrightarrow{g} T$ are maps of DGAs such that f makes S into a semi-free $R-$module and for some $n \in \mathbb{Z}$, $Hom_R(S,R) \cong \Sigma^n S$. In addition suppose, there exists $m \in \mathbb{Z}$ such that for the semi-projective resolution of T as a $S-$module $\hat{T}$, $Hom_S(\hat{T},S) \sim \Sigma^m \hat{T}$. Then for the semi-projective resolution of T as an $R-$module, $\hat{\hat{T}}$, we have*

$$Hom_R(\hat{\hat{T}},R) \sim \Sigma^{n+m}\hat{\hat{T}}.$$

Proof. First of all, as S is semi-free over R then every semi-projective $S-$module is a semi-projective $R-$module as well. Hence $\hat{T}$ can be considered as the semi-projective resolution of T as an $R-$module as well and therefore $\hat{\hat{T}} \sim \hat{T}$. The following series of homotopy equivalences and isomorphisms lead to the desired result.

$$\begin{aligned} Hom_R(\hat{\hat{T}},R) &\sim Hom_R(\hat{T},R) \\ &\cong Hom_R(\hat{T} \otimes_s S, R) \\ &\cong Hom_S(\hat{T}, Hom_R(S,R)) \\ &\cong Hom_S(\hat{T}, \Sigma^n S) \sim \Sigma^{n+m}\hat{T} \end{aligned}$$

□

Proposition 4.35. *Let G be a compact connected Lie group, H be a closed subgroup and G/H be an orientable manifold.*

i. If $rankG = rankH$ then

$$Hom_{H^*(\mathbf{B}G)}(H^*(\mathbf{B}H), H^*(\mathbf{B}G)) \cong \Sigma^{-dim(G/H)}H^*(\mathbf{B}H).$$

ii. If H is any torus inside G then

$$Hom_{H^*(\mathbf{B}G)}(\widehat{H^*(\mathbf{B}H)}, H^*(\mathbf{B}G)) \sim \Sigma^d \widehat{H^*(\mathbf{B}H)}.$$

Proof. (i) By 4.32 we have

$$H^*(\mathbf{B}H) \cong H^*(\mathbf{B}G) \otimes_{\mathbb{Q}} H^*(G/H)$$

and as $H^*(\mathbf{B}H)$ is a free finitely generated module its Betti table as $H^*(\mathbf{B}G)-$module is exactly like the Betti table of $H^*(G/H)$. Since G/H is an orientable manifold its Betti table is symmetric and therefore (i) holds.

(ii) Suppose T_G is the maximal torus of G and consider the following sequence of DGAs maps

$$H^*(\mathbf{B}G) \longrightarrow H^*(\mathbf{B}T_G) \longrightarrow H^*(\mathbf{B}H)$$

induced by inclusions

$$H \longrightarrow T_G \longrightarrow G\ .$$

Since $rankG = rankT_G$ then (i), Lemma 4.34 and Example 4.30 yield the result. To be more precise $d = -dim(G/T_G) - dim(T_G/H)$. $\square$

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